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Intraclass and Interclass Correlation Coefficients with Application

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Abstract

In the paper a review and a comparison of estimates and tests of intraclass and interclass correlation coefficients is presented.

1 Introduction

Rosner (1982) pointed out that the basic unit for statistical analysis in ophthalmological studies is the eye rather than person. Sometimes, one eye is used as the treated eye and the other as the control. In this case standard methods of estimation and hypothesis testing are valid. But if the purpose is to compare two different types of people on some finding in an ocular examination such as a comparison of intraocular pressures in persons in different age groups, their values from the two eyes are highly correlated and the above statistical methods are not valid.

One of the appropriate methods for such a design is given by a nested mixed effects analysis of variance (ANOVA) and appropriate extensions to multivariate methods in ophthalmology with application to other paired-data situations.

The methods have implications for otolaryngological data, dental data as well as in the analysis of familial data.

An assessment of the degree of resemblance among family members with respect to some biological or psychological attribute such as weight, differences in the skin, arm length or blood pressure, is of particular interest to geneticists.

1.1 Intraclass correlation model in ophthalmologic studies

Suppose we have g groups of persons and we wish to compare them with respect to some ocular finding y (for example intraocular pressure). This is presented by Rosner (1982). The model used is given by

$$Y_{ijk} = \mu + \alpha_i + \beta_{ij} + e_{ijk} \qquad i = 1,\ldots,g; \quad j = 1,\ldots,P_i; \quad k = 1,\ldots,N_{ij}$$

where there are P_i persons in the ith group $i = 1,\ldots,g$; $P = \sum P_i$ and each person contributes N_{ij} eyes, $N_{ij} = 1$ or 2, $i = 1,\ldots,g$; $j = 1,\ldots,P_i$; $\beta_{ij} \sim N(0,\sigma_\beta{}^2)$ is the

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effect of persons within groups, $e_{ijk} \sim N(0,\sigma^2)$ is the effect of eyes within persons which considered to be random effect, μ and $\{\alpha_i\}$ are constants.

But there exists a difficult problem due to the unbalanced nature of the design whereby different persons contributed either one or two eyes to the analysis. A reasonable approximate method for this problem is given by (Searle, 1971:367).

The test procedure is to test the hypothesis H_0 : all α_i are equal versus H_1 : some of the α_i are unequal.

The test statistic $\lambda = MSG/MSP$ which follows an $F_{g-1,P-g}$ distribution under H_0 , where MSG is mean square between groups and MSP is mean square between eyes within persons and appropriate intraclass correlations ρ is given by $\rho^* = \sigma_\beta^{*2}/(\sigma_\beta^{*2} + \sigma^{*2})$ where $\sigma_\beta^{*2} = \max\{0, MSP - (MSE/N)\}$, $\sigma^{*2} = MSE$.

But if we assume that two eyes from the same person are independent random variables then we have an ordinary one-way ANOVA model.

1.2 Constant R method

Rosner (1982) assumed this model where $Y_{ijk} = 1$ if the k th eye of the j th person in the i th group is affected, and 0 otherwise, $i = 1,\ldots,g$; $j = 1,\ldots,P_i$; $k = 1,2$ and $P(Y_{ijk} = 1) = \lambda_i$, $P(Y_{ijk} = 1|Y_{ij(3-k)} = 1) = R\lambda_i$ for some positive constant R. The constant R is a measure of dependence between two eyes of the same person.

The primary aim is to test $H_0 : \lambda_1 = \lambda_2 = \ldots = \lambda_g = \lambda$ versus $H_1 : \lambda_r \neq \lambda_s$ for at least one pair (r,s).

Rosner estimated "the effective number of eyes per person" under this model by

$$e^* = \frac{2\lambda^*(1-\lambda^*)}{\lambda^*(1-\lambda^*) + (R^*-1)\lambda^{*2}}$$

Let P_{ij} denote the number of persons in the i th group with j affected eyes, $i = 1,\ldots,g$; $j = 0,1,2$ then

$$\lambda^* = \frac{\sum(P_{i1} + 2P_{i2})}{2N},$$

$$R^* = \frac{4N\sum P_{i2}}{(\sum P_{i1} + 2\sum P_{i2})^2}$$

An appropriate test is given by

$$T = \frac{e^*}{\lambda^*(1-\lambda^*)}\sum P_i(\lambda_i^* - \lambda^*)^2.$$

Rosner presented the extensions of these methods such as multiple regression method and he related the value of a normally distributed outcome variable Y_{ij} for the j th subunit of the i th primary unit $(i = 1,\ldots,n;\ j = 1,\ldots,t_i)$ to the values of k independent variables $X_{ij1},\ldots,X_{ijk}$, where X_{ijk} denotes the k th independent variables for the j th subunit of the i th primary unit:

$$Y_{ij} = \beta_0 + \sum \beta_k X_{ijk} + e_{ij}$$

where $var(e_{ij}) = \sigma^2$, $\rho(e_{ij}, e_{ik}) = \rho$; $i = 1,\ldots,n$; $j \neq k = 1,\ldots,t_i$ and our aim is to test the hypothesis $H_o : \beta_k = 0$, all other $\beta_i \neq 0$ versus H_0 : all $\beta_i \neq 0$.

2 Estimation of the degree of resemblance among family members

One of the aims of the analysis of familial data is to estimate the resemblance among the members of a family. Usually we face two distinct problems:

- to estimate the resemblance among the children themselves, the problem of estimation of intraclass correlation
- to estimate the resemblance among parents and their children, the problem of estimation of interclass correlation

3 Estimation of intraclass correlation

We denote the score of ith family's offspring by X_{ij} $i = 1, \ldots, k$; $j = 1, \ldots, n_i$, where k is the total numbers of families studied and n_i is the position in the family and $N = \sum n_i$ is the total number of observations.

We assume that size of family offspring are not necessarily the same in each family, since that problem has been most frequently encountered in practice.

3.1 Method of components of variance

Donner (1979) suggested to use analysis of variance:

$$X_{ij} = \mu + a_i + e_{ij}$$

where μ is the grand mean of all observations in the population, the family effects $\{a_i\}$ are identically distributed with mean 0 and variance σ_A^2 , the residual errors $\{e_{ij}\}$ are identically distributed with mean 0 and variance σ_e^2 and $\{a_i\}$, $\{e_i\}$ are completely independent. The variance of X_{ij} is given by $\sigma_X^2 = \sigma_A^2 + \sigma_e^2$.

The proportion of the variation among groups is also known as

$$\rho_{cc} = \frac{\sigma_A^2}{\sigma_A^2 + \sigma_e^2}$$

Since family size n_i differs among families it is obvious that no single value of n_i would be appropriate in the formula. We therefore use an average n_i ; this is not a simple $\bar{n}$, the arithmetic mean of the n_i's but

$$n_o = \bar{n} - \frac{1}{(k-1)N} \sum_{i=1}^{k} (n_i - \bar{n})^2$$

which is an average usually close to, but always less then $\bar{n}$, unless sample size are equal when $n_o = \bar{n}$.

Since unbiased estimates of σ_e^2 and σ_A^2 are given by $S_W^2 = MSW$ and $S_A^2 = (MSA - MSW)/n_o$ the analysis of variance intraclass correlation coefficient as

$$r_A = \frac{S_A^2}{S_A^2 + S_W^2} \cdot \frac{MSA - MSW}{MSA - MSW + n_o MSW} = \frac{F-1}{n_o + F - 1}$$

where $F = \frac{MSA}{MSW}$.

3.2 Common correlation model called also random effects model

The other model is called "common correlation model" in which the observations X_{ij} are assumed to be distributed about the same μ and with same variance σ^2 in such a way that X_{ij} and X_{il} in the same class have a common correlation ρ.

3.3 Maximum likelihood estimator

Suppose we have

$$X_i = (X_{i1}, \ldots, X_{in_i}); \quad X_i \sim N(\mu_i, \Sigma_i), \quad i = 1, \ldots, k$$

$$\mu_i = (\mu, \ldots, \mu), \quad \Sigma_i^{jl} = \rho\sigma^2, \quad \Sigma^{jj} = \sigma^2, \quad j, l = 1, \ldots, n_i, \quad j \not\equiv l$$

$$L(X_1, \ldots, X_k|\mu, \sigma^2) = (2\pi)^{-N/2} \prod_{i=1}^{k} |\Sigma_i|^{-1/2} \exp\{-\frac{1}{2} \sum_{i=1}^{k} (X_i - \mu_i)^T \Sigma_i^{-1} (X_i - \mu_i)\}$$

$$-2 \ln L = N(1 + \ln \sigma^2 + \ln(2\pi)) + (N - k) \ln(1 - \rho) + \sum_{i=1}^{k} \ln W_i$$

where $W_i = 1 + (n_o - 1)\rho$.

This expression may be numerically minimized with respect to ρ to yield the maximum likelihood estimation r_M.

It is interesting to note that for the case $n_i = n$, $i = 1, \ldots, k$ we have $r_M \equiv r_P$ Pearson's correlation coefficient which is expressed by

$$r_P = \frac{1}{Nn(n-1)S_x^2} \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j \not\equiv l}}^{n} \sum_{l=1}^{n} (X_{ij} - \bar{X})(X_{il} - \bar{X})$$

where $\bar{X}$ and S_x^2 are mean and variance.

If one remembers the last research work (for example among the siblings for the blood pressure) where $\rho_{cc} < 0.5$ (small value) and in the cases when we do not know anything about intraclass correlation coefficient, in such case, we use the model of "common correlation", i.e. r_A.

Under the assumption that ρ_{cc} has a large value (0.8) we use the maximum likelihood estimator.

4 Estimation of interclass correlation coefficients

Let us take for example a sample of measurements from k families and $X_{i0}, X_{i1}, \ldots, X_{in_i}$ representing measurements from i th family where X_{i0} is the mother's score (parent's score in general) and $X_{i1}, \ldots, X_{in_i}$ are the scores of her n_i siblings.

Suppose we have

$$X_i = (X_{i0}, X_{i1}, \ldots, X_{in_i}) \sim N(\mu, \Sigma), \quad i = 1, \ldots, k$$

$$\mu_i = (\mu_m, \mu_c, \ldots, \mu_c), \quad \Sigma_i^{1j} = \Sigma_i^{j1} = \rho_{mc}\sigma_m\sigma_c, \quad j = 1, \ldots, n_i$$

$$\Sigma_i^{jj} = \sigma_c^2, \quad \Sigma_i^{jl} = \rho_{cc}\sigma_c^2, \quad j, l = 1, \ldots, n_i; j \neq l$$

.

The intraclass correlation is denoted by ρ_{cc} and the mother sib interclass correlation is denoted by ρ_{mc}.

4.1 Pairwise estimator

This method is performed by pairing each mother's score with each of her sibling's scores and considering the collection of all such pairs over all families.

If we consider each of the pairs (X_i, X_{ij}) as an independent observation from a bivariate normal distribution $N(\lambda, \Sigma)$ where

$$\lambda = (\mu_m, \mu_c), \qquad \Sigma = \begin{bmatrix} \sigma_m^2 & \rho_{mc}\sigma_m\sigma_c \\ \rho_{mc}\sigma_m\sigma_c & \sigma_c^2 \end{bmatrix}$$

then

$$\rho_{mc} = \frac{\sum_{i=1}^{k}(X_{i0} - \bar{X}_m)\sum_{j=1}^{n_i}(X_{ij} - \bar{X}_c)}{\sqrt{\sum_{i=1}^{k} n_i (X_{i0} - \bar{X}_m)^2}\sqrt{\sum_{i=1}^{k}\sum_{j=1}^{n_i}(X_{ij} - \bar{X}_c)^2}}$$

where

$$\bar{X}_m = \frac{\sum_{i=1}^{k} n_i X_{i0}}{\sum_{i=1}^{k} n_i}, \quad \bar{X}_c = \frac{\sum_{i=1}^{k}\sum_{j=1}^{n_i} X_{ij}}{\sum_{i=1}^{k} n_i}$$

.

Each dependence arises for two reasons: the same mother will appear in as many pairs as children possessed and there will be positive correlation among children within the same family, i.e. $\rho_{cc} > 0$.

4.2 Sib-mean estimator

Falconer (1960) recommends pairing each mother's score (X_{i0}) with the mean of her sibling's scores, i.g. $\bar{X}_{ic} = \sum_{j=1}^{n_i} X_{ij}/n_i$

$$\rho_m^* = \frac{\sum_{i=1}^{k}(X_{i0} - \tilde{X}_m)(\bar{X}_{ic} - \tilde{X}_c)}{\sqrt{\sum_{i=1}^{k}(X_{i0} - \tilde{X}_m)^2}\sqrt{\sum_{i=1}^{k}(\bar{X}_{ic} - \tilde{X}_c)^2}}$$

where

$$\tilde{X}_m = \frac{1}{k}\sum_{i=1}^{k} X_{i0}, \quad \tilde{X}_c = \frac{1}{k}\sum_{i=1}^{k} \bar{X}_{ic}$$

.

This estimator depends on sample size.

4.3 Random-sib estimator

Biron (1977) selected one child randomly from those families having two or more children and computed the parent-child correlation on the basis of the resulting subsample only.

$$\rho_r^* = \frac{\sum_{i=1}^{k}(X_{i0} - \bar{X}_m)(X_{ij}^* - \bar{X}_c^*)}{\sqrt{\sum_{i=1}^{k}(X_{i0} - \bar{X}_m)^2}\sqrt{\sum_{i=1}^{k}(X_{ij}^* - \bar{X}_c^*)^2}}$$

where X_{ij}^* denotes a random sibling from the i th family and

$$\bar{X}_c^* = \frac{1}{k}\sum_{i=1}^{k} X_{ij}^*$$

4.4 Ensemble estimator

One solution to this problem (Rosner, Donner 1977) is to compute the expected value of ρ_r^* over all possible selections of members over all families. This solution would be unwieldy, since it would require the computation of $\prod_{i=1}^{k} n_i$ distributed correlations. But using some approximations we have

$$\rho_e^* = \frac{\sum_{i=1}^{k}(X_{i0} - \bar{X}_m)(\bar{X}_{ic} - \bar{X}_c)}{\sqrt{\sum_{i=1}^{k}(X_{i0} - \bar{X}_m)^2}\sqrt{\sum_{i=1}^{k}\sum_{j=1}^{n_i}(X_{ij} - \bar{X}_{ic})^2/n_i(1-\frac{1}{k}) + \sum_{i=1}^{k}(\bar{X}_{ic} - \bar{X}_c)^2}}$$

5 Discussion

(Donner, Rosner 1977) compared the pairwise, sib-mean, random-sib and ensemble estimators using, as the criterion of comparison, their mean square errors obtained from Monte Carlo simulation.

The pairwise estimate has a lower mean square error than either sib-mean square errors, which we utilized to compare the estimators under performing 100 iterations. The pairwise estimator is more effective than ensemble estimator at low values of ρ_{cc} and the ensemble estimator is more effective at high values of ρ_{cc}. For intermediate values of ρ_{cc}, the two estimator perform about equally well.

These methods can also be applied to other types of paired data, as in matched studies with a variable matching ratio, where one has a continuous outcome variable and wishes to control for other confounding variables while maintaining the matching.

References

[1] Donner A. (1978): The number of families required for detecting the familial aggregation of a continuous attribute. *American Journal of Epidemiology,* **108**, 425-471.

[2] Donner A. (1979): The use of correlation and regression in the analysis of familial resemblance. *American Journal of Epidemiology*, **110**, 335-342.

[3] Donner A. and Koval J.J. (1980): The estimation of intraclass correlation in the analysis of familial data. *Biometrics*, **36**, 19-25.

[4] Falconer, D.S. (1960): *Introduction to Quantitative Genetics*. London: Longman Groups.

[5] Rosner B., Donner A., and Hennekens C.H. (1977): Estimation of interclass correlations from familial data. *Applied Statistics*, **26**, 179-187.

[6] Rosner B. and Donner A. (1979): Significance testing of interclass correlations from familial data. *Biometrics*, **35**, 461-471.

[7] Rosner B. (1982): Statistical methods in ophthalmology: An adjustment for the intraclass correlation between eyes. *Biometrics*, **38**, 105-114.

[8] Rosner B. (1984): Multivariate methods in ophthalmology with application to other paired-data situations, *Biometrics*, **40**, 1025-1035.

[9] Elston, R.C. (1975): On the correlation between correlations. *Biometrika*, **62**, 133-140.