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# Stochastic Growth Models

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**Abstract**

Our main interest in this article will be the limit shape in a two-dimensional stochastic growth model. This model can represent spreading of fire (or virus, information, etc.) on a fixed lattice. Limit shape depends on various parameters: the shape and size of chosen neighbourhood and the excitability of the media.

First we define what a growth model is, then we focus on one kind of such a model. We derive a certain equation in one dimension which tells us the speed of a spreading. Then we use Wulff transformation to determine the limit shape in two dimensions. This method is used only for square neighbourhood, but minor modifications make it work for other shapes. However, technical problems do occur. At the end we consider possible interesting modifications to the system.

## 1 Introduction

A lot of real-life processes resemble a model where either sets or values belonging to some points grow. First models of such nature were formulated for spreading of a forest fire (or a spreading of a piece of information, or in medical community, spreading of a disease). In this area there are a lot of already known (and mathematically proven) results. We will state the basic known facts and try to develop some new facts in this and other kinds of growth models.

First let us state a universal scheme which can describe a growth model:

*i.* The first thing we need is a population. We denote it by $V$. From our point of view $V$ will usually be equal to $\mathbb{Z}^d$, $d$ being the dimension (in most cases we will be interested in $d = 2$).

*ii.* Every point in $V$ (an individual, either a tree or a person) can be in any of the states $s \in S$. The state space $S$ is (partially) ordered with $\leq$. Further demands on $(S, \leq)$ will be determined for every model. In case of a forest fire $S = \{0, 1\}$, where 0 means the tree is alive and 1 means the tree is burning, or $S = \{0, 1, 2\}$, where 0 and 1 as before and 2 means the tree is gone. In both cases the ordering is the natural ordering $0 \leq 1 \leq 2$.

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*iii.* The evolution of the process is described by a function $f : V \times \mathbb{R}^+ \to S$ with $f(v, \cdot)$ being an increasing function. This is of course a random function and its distribution is almost impossible to determine.

*iv.* The main part of any model is its stochastic dynamics, i.e., what kind of rules apply to $f$. This actually determines the distribution of $f$.

*v.* The questions we want answers to can be formulated in such a way that we must evaluate $\Phi(f)$, where $\Phi$ is some functional. For example, if we want to calculate the probability that in a forest fire every tree burns down, then $\Phi$ equals

$$\Phi(f) = P(\lim_{t\to\infty} f(x,t) = 1 \text{ for every } x).$$

This model includes the model where the set of (specific) points increases in time. If we denote the set of occupied points in time $t$ by $A(t)$ then we want $A(t)$ to be an increasing sequence in $t$. So we define a function $f$ with $f(x,t) = 0$ meaning $x \notin A(t)$ and $f(x,t) = 1$ meaning $x \in A(t)$. So $A(t) = \{x; f(x,t) = 1\}$.

# 2 Examples

The first example we will examine is a model of formation of a mountain range. We need two dimensions for our base and another for the height. Let $V = \mathbb{Z}^2$ and let the state space be $S = \mathbb{Z}^+$. The time will be discrete and let the growth dynamics be given by

$$f(v,t) = \begin{cases} f(v,t-1)+1, & \begin{array}{l} f(v+x,t-1) > f(v,t-1) \\ \text{for some } x \in \mathcal{N} \end{array} \\ f(v,t-1) + X_{v,t-1}, & \text{otherwise} \end{cases}$$

where $\mathcal{N}$ is the neigbourhood of four adjacent points, e.g., the neighbourhood of origin $(0,0)$ are the points $(-1,0)$, $(0,1)$, $(1,0)$, and $(0,-1)$. $X_{v,t-1}$ are independent Bernoulli random variables with parameter $p_v$ for all $v$, $t$. That means $P(X_{v,t} = 1) = p_v$ and $P(X_{v,t} = 0) = 1 - p_v$. We start with a flat land, i.e., $f(v,0) = 0$ for all $v$.

The parameters $p_v$ tell us how fast the landscape will rise above point $v$. Because of the growth dynamics some points will be pulled by neighbouring peaks that rise faster. The maximum angle of the slope is $\frac{\pi}{4}$ (looking in direction of axis $x$ and $y$). The question we can ask is how the landscape will look like after a long time, depending on the distribution of $p_v$. Will we get a lot of dominating peaks, or will there be only a few peaks?

Figure 1 shows the landscape at time $t = 60$. The parameter $p_v$ is uniformly distributed on $[0, \frac{1}{2}]$.

We can already see the forming of some peaks, and especially an area where the landscape is flatter and an area where we have a mountain range. In the long run the point $v$ with the highest parameter $p_v$ will prevail and all other points will just follow it in their growth. Of course, there will be small fluctuations which are

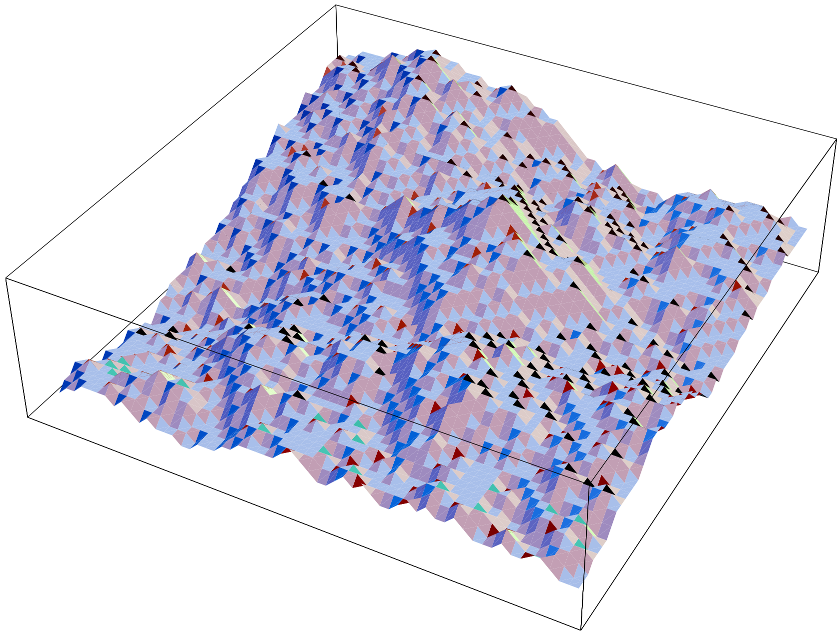


**Figure 1:** Growth model at time $t = 60$.

(in limit, usually) normally distributed. So another question is about the order of magnitude of these fluctuations.

Little is known about this model so far. Some interesting results have been developed in the case of one dimension where the peaks pull up the points on one side only – so the point grows only if its left neighbour is higher but can stay the same if its right neighbour is higher.

In the following, we will focus ourselves on one specific model.

# 3 Specific growth model

Our main interest in this article will be the next model. The population will be all the integer points in $d$-dimensional space. Our special interest will be $d = 2$.

Let the state space be $S = \{0, 1\}$, so every particle is either of type 0 or type 1. Particles in state 1 are usually called excited while particles in state 0 are sometimes called dormant. So every particle will spend some time in its dormant phase and at some time, possibly $\infty$, it will become excited. Such a jump occurs when certain conditions considering neighbours of an individual point are fulfilled. The rules of dynamics, formulated through the function $f$, are as follows:

The first rule says that every excited point stays excited forever, i.e., if $f(v, t) = 1$, then $f(v, t') = 1$ for all $t' > t$.

Second, the point does not become excited if too few points in its neighbourhood $\mathcal{N}$ are excited. More precisely, if $f(v, t) = 0$ and $\sum_{x \in \mathcal{N}} f(v + x, t) < \vartheta$, then (in case of continuous time) there exists $\varepsilon > 0$ such that $f(v, t + \varepsilon) = 0$ and (in case of discrete time) $f(v, t + 1) = 0$.

The third rule states that if $v$ is not excited but has enough excited neighbours it can become excited. Here we have two possibilities regarding the nature of time: if time is discrete, $v$ will get excited in the next unit of time with some probability $p_v$. If time is continuous, $v$ will get excited at rate $\alpha_v$ meaning that the time before $v$ gets excited will be an exponential random variable with parameter $\alpha_v$.

We will be interested in the existence of a limit shape. Let

$$A(t) = \{v \in V; f(v,t) = 1\}.$$

Then, under some regularity conditions on neighbourhood $\mathcal{N}$, next theorem holds:

**Theorem 1.**

$$\frac{A(t) + [-\frac{1}{2}, \frac{1}{2}]^d}{t} \xrightarrow{\text{a.s.}} U, \tag{3.1}$$

where $U$ is some fixed convex set.

Theorem 1 is known as the shape theorem.

Let $\mathcal{N}$ be a square neighbourhood in dimension $d = 2$ (which we will inflate and call $n \cdot \mathcal{N} = [-n, n] \times [-n, n]$), and the threshold $\vartheta$ will be equal to the proportion $\lambda$ of area of half of a square $n \cdot \mathcal{N}$ so $\vartheta = \lambda \cdot n^2$. The limit set $U$ from Theorem 1 depends on parameter $\lambda$ and neighbourhood $n \cdot \mathcal{N}$ so we should write it in the form $U_{n,\mathcal{N},\lambda}$. We fix the parameters $\lambda$ and $\mathcal{N}$ (and can therefore omit them) and take special interest in the limit

$$\frac{U_n}{n} \xrightarrow{\text{H}} U^*,$$

if it exists. H denotes convergence in Hausdorff metrics.

**Example.** $n = 5$, $\vartheta = 1$, $\mathcal{N} = [-1, 1]^2$. Black points are points in state 1, the gray area represents the points which are in the neighbourhood of one of the points in state 1 and are thus waiting to jump to state 1. The number of points in state 1 would grow exponentially if there would be no spatial limitations because every point in state 1 would make a certain amount of points jump from state 0 to state 1. In only $d$-dimensions we get polynomial growth of order $d$ so the width of the shape grows linearly.

To solve the two–dimensional problem we will first solve the same problem in one dimension. Nothing changes in the model, except that $V = \mathbb{Z}$.

# 4 The one–dimensional case

For large $n$, let $f$ be the density profile of excited particles at the wave front spreading to the left, rescaled in such a way that the one–sided neighbourhood is equal to the interval $[0, 1]$. So $f(0, t) = 0$, and the interval $[0, \vartheta]$ is rescaled to $[0, 1]$.

After a tick of time $f(x, t)$ changes according to dynamics described in previous section. The value of $f$ in a time difference $\Delta t$ equals to

$$f(x, t + \Delta t) = f(x - \Delta x, t) + (1 - f(x, t))\Delta t$$

From this equation we derive a partial differential equation

$$f_t(x, t) = -\frac{1 - \lambda}{f(1, t)} f_x(x, t) + 1 - f(x, t) \tag{4.1}$$

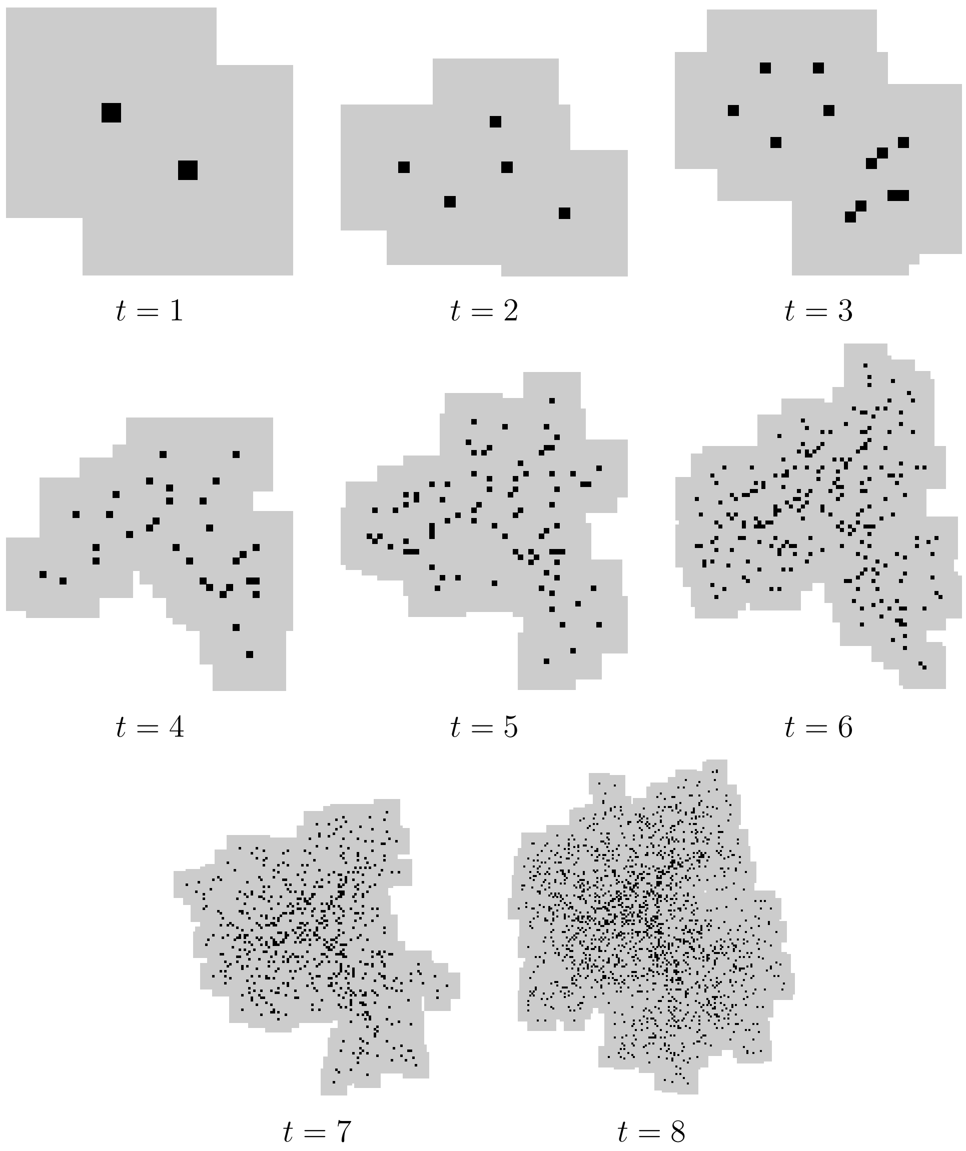



**Figure 2:** Growth model in continuous time.

with the condition

$$\int_0^1 f(x,t)dx = \lambda. \tag{4.2}$$

This is actually a kind of conservation law PDE since integral in (4.2) is conserved in time. The equation (4.1) is however not typical because of the term $f(1,t)$. In equilibrium we get a differential equation

$$\frac{1-\lambda}{f(1)} f' = 1 - f,$$

yielding

$$f(x) = 1 - e^{-x/w},$$

with $w$ being the speed of spreading. Because $\int_0^1 f(x)dx = \lambda$, $w$ is the solution of

$$1 - \lambda = w(1 - e^{-\frac{1}{w}})). \tag{4.3}$$

If $\lambda$ is close to 1, i.e. if nearly the whole neighbourhood must be excited before a particle can jump to excited state, the growth will be slow – this phase is called the quenched phase – and $w$ will be close to 0. The equation (4.3) becomes

$$1 - \lambda = w.$$

With $\lambda$ close to 0 the system grows very quickly and the phase is called the annealed phase. The equation (4.3) now reads

$$1 - \lambda \approx w(1 - (1 - \frac{1}{w} + \frac{1}{2w^2})) = 1 - \frac{1}{2w},$$

from where we get $w \approx \frac{1}{2\lambda}$.

We can verify our result through simulation. Let the neighbourhood be 2000 points wide and let us compute the local density by averaging over 100 closest points. Let $\lambda = 0.8$. After some time (approximately after 2000 jumps from 0 to 1) we get the density profile shown on the left graph of Figure 3. On the right graph there is the limit solution in equilibrium. So the convergence (and its speed) is nicely illustrated.

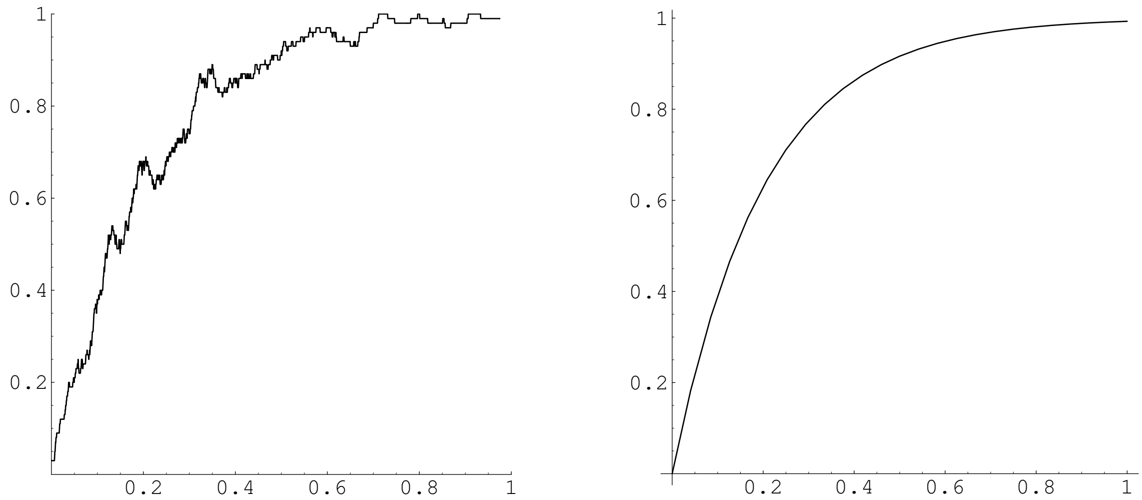



**Figure 3:** Whatever the starting conditions are the density of particles converges to equilibrium.

# 5 The two–dimensional case

In two dimensions we start by determining the speed of spreading in every direction. For a fixed direction, defined by the angle of spreading, the problem is reduced to one-dimensional case. However not every point in this one dimension has the same weight. This is because the square is not symmetrical with respect to the axis determined by the direction of spreading. The weight of every point on the axis is proportional to the maximum length of a line in a square orhogonal to the direction at that point. Incorporating different weights of points into our model the modified equation (4.3) becomes

$$2 - \lambda = 2w\sqrt{1+\alpha^2} + \frac{1+\alpha^2}{\alpha} e^{-\frac{1+\alpha}{w\sqrt{1+\alpha^2}}} \left(1 - e^{\frac{2\alpha}{w\sqrt{1+\alpha^2}}}\right) w^2, \tag{5.1}$$

where $\alpha = \tan\varphi$ and $\varphi$ is the angle of spreading. $w$ is the velocity of spreading in that direction. The resulting speed is a function of the direction of spreading, let us say $w = w(\varphi)$. Unfortunately, the limiting shape is not simply

$$K_w = \{rw(\varphi)\mathbf{u}(\varphi); r \in [0,1], \varphi \in [0, 2\pi)\},$$

as one would expect. $\mathbf{u}(\varphi)$ is the unit vector in direction $\varphi$. This set is not necessarily convex but the limiting shape must be convex. In case of a square neighbourhood the set $K_w$ is allways convex (because every line through the center divides it into two halves with equal area) but in case of a triangle neighbourhood $K_w$ can be not convex.

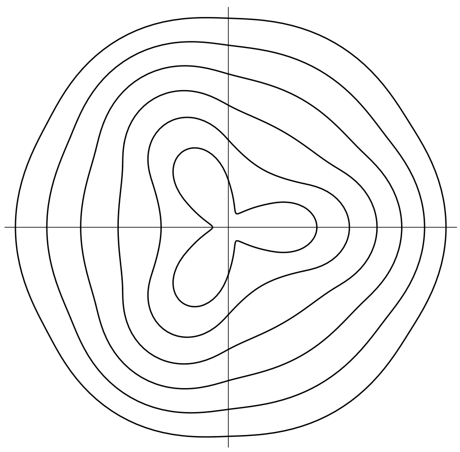


**Figure 4:** The curve $w(\varphi)$ for a triangle neighbourhood for different values of $\lambda$ (not on the same scale), the innermost curve corresponds to the largest $\lambda$.

Obviously the right limiting shape is the set of points $\mathbf{x}$ for which the following condition on scalar products holds:

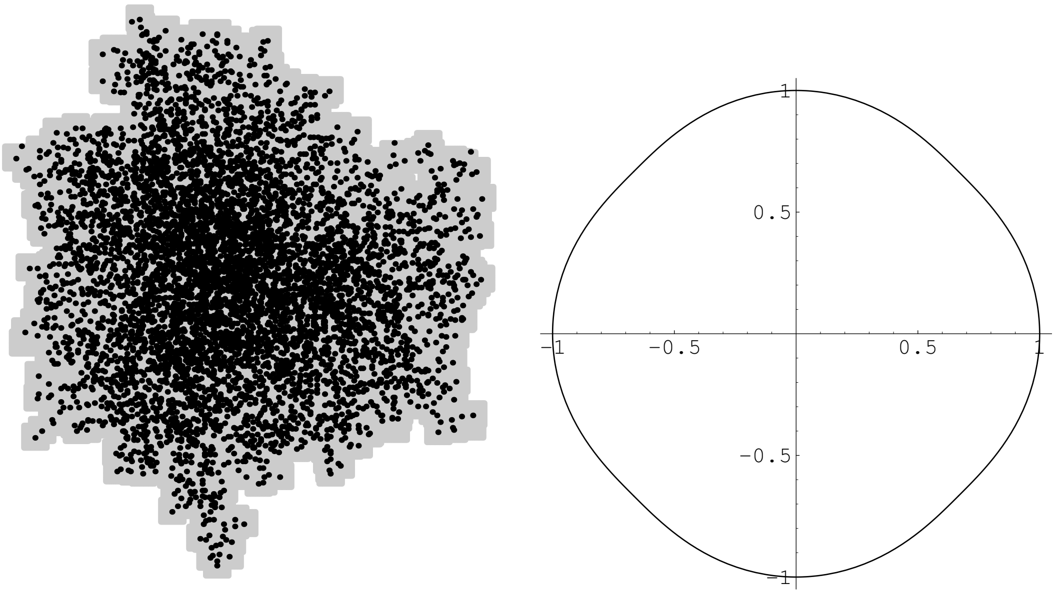
$$\mathbf{x} \cdot (w(\varphi)\mathbf{u}(\varphi)) \leq (w(\varphi)\mathbf{u}(\varphi)) \cdot (w(\varphi)\mathbf{u}(\varphi)) \text{ for all } \varphi.$$

The limit shape then equals the set $K^*_{1/w}$ where $A^* = \{\mathbf{x}; \mathbf{a} \cdot \mathbf{x} \leq 1 \text{ for all } \mathbf{a} \in \mathrm{A}\}$. This concept is known as the Wulff construction. So even if $K_w$ is convex it is not equal to the boundary of the limit shape (except in the case when it is a $L^2$ ball).

When $\lambda \to 0$, $w$ is of the order $\frac{1}{\lambda}$. The renormalized shape of the excited area is bounded by the curve

$$x^2 + y^2 = \frac{2x^2}{3x - 1}$$

for $x \geq \frac{2}{3}$; the whole curve is obtained by rotating this curve by $\frac{n\pi}{2}$, $n = 1, 2, 3$.




*Far from limit* *Equilibrium*

**Figure 5:** Convergence in $\mathbb{Z}^2$. If $\lambda$ is small, the occupied set grows rapidly and the convergence to the limit is slower.

This shape resembles the circle. However, it is not the circle. Maybe we were expecting the limit shape to be a circle because a small $\lambda$ means a dynamics without many conditions. Much like throwing an object into the water causes circular waves no matter what the shape of the object is. In our model this is not the case because when a point jumps from state 0 to state 1 it can be any point from the "infected" area, and this area should roughly resemble the shape of the neighbourhood. By this thinking the limiting shape should be equal to the chosen neighbourhood, namely a square. But it is not, this time because the infected area is of the same shape as the neighbourhood only at the beginning of time after which it changes. This consideration works for discrete time where the limit shape actually is equal to the square. So in limit we get the shape between our first two guesses.

This is a instructive case: we should not trust our instinct blindly.

Lastly we take a look at different limiting shapes for square neighbourhoods in two dimensions but for different parameters $\lambda$. The smallest shape shows the case when $\lambda$ is near to 2 and the largest one when $\lambda$ is near to 0.

# 6 Further questions

Let us consider possible generalizations of our model.

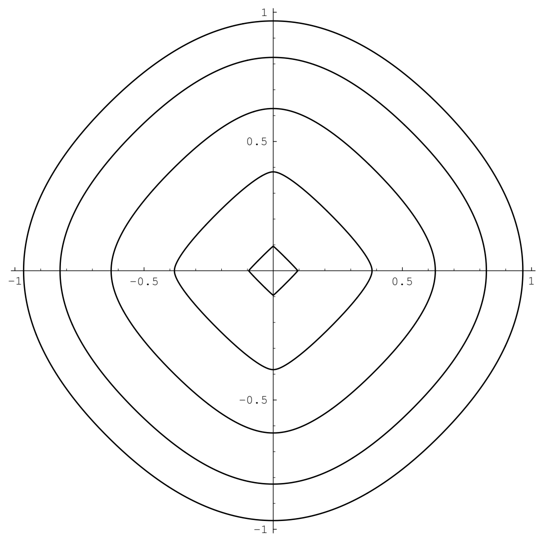



**Figure 6:** Sets $K_w$ for $\lambda = 0.1, 0.5, 1, 1.5, 1.9$ (from inside out) each multiplied by $\lambda$.

The first question is what happens in dimensions higher than 2. This question is easy to answer. All the methods that we used in the 2-dimensional case work also in 3 or more dimensions. The only problem in higher dimensions is exactly the same as in 2 dimensions - the problem of forming of holes. In higher dimensions the tentacles form at the same rate as in 2 dimensions but they rarely intersect, therefore this is actually not a problem. So we get a speed of spreading into every direction in space, and the limit shape can again be put together through a Wulff transformation. If we solve a 2-dimensional problem, we will probably be able to solve it in dimensions higher than 2 as well. We can therefore limit ourselves to dimension 2, which is actually quite a natural dimension for our considerations.

The next thing we consider is what changes if the neighbourhood is not a square. Of course, it still has to be convex – otherwise the limit theorem does not apply. Again, all the methods work. If the neighbourhood $\mathcal{N}$ equals a circle, all the limit shapes will be circles, because of symmetry. Things are different if the neighbourhood $\mathcal{N}$ equals a triangle. In that case the curve (in polar coordinates) $r(\varphi) = w(\varphi)$ is usually not convex so again we have to apply Wulff transformation to get the limiting shape.

Figure 7 represents the limiting shape when $p$ is small and the threshold $\theta$ is close to the maximum possible threshold, i.e., the spreading is very slow. The light triangle represents the neighbourhood $\mathcal{N}$, the curve is the speed of spreading ($r(\varphi) = w(\varphi)$) and the dark area is the limit shape. The picture tells us that in certain directions the spreading is rapid (in the directions where the neighbouring triangle is narrow) whereas in other directions there is hardly any movement, due to a relatively small area of the triangle neighbourhood behind the wave front. The result is the limiting shape which is similar to a triangle.

All our work so far has been done in continuous time. Do the results change if time is discrete? Again the dimension to start with is dimension one. After some

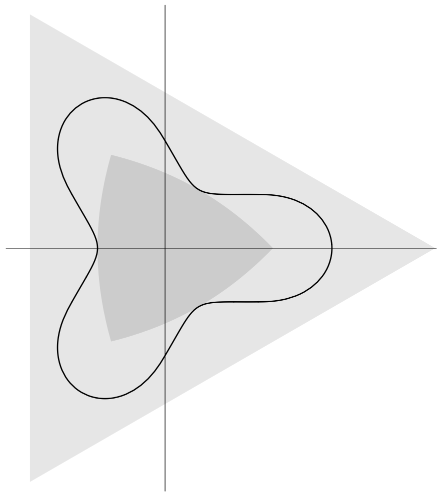


**Figure 7:** Limiting shape when neighbourhood $\mathcal{N}$ equals a triangle.

consideration, we can conclude that the limiting profile is not a continuous function but is a step function. All the steps are of the same length and this length is at the same time the speed of spreading. Something very similar must happen in higher dimensions. The computation of the limit shape is even more complicated than in continuous time.

The next interesting question is what happens if we let particles loose so that they can move around in some space, for instance in $\mathbb{R}^d$. This is again a very natural assumption, maybe not so in the case of a forest fire but definitely in the case of an epidemic. Here we encounter lots of problems. The first is what movement is allowed and how to describe it. If we let every particle move according to an independent Brownian motion with small variance (compared to the speed of spreading) nothing will happen to the solution because we had small normal fluctuations at the border of the infected area in our original model, and that contributed nothing to the limiting shape. But if particles have long range of traveling then maybe we can change the model by changing the nature of neigbourhood $\mathcal{N}$. But the particles in our new neighbourhood $\mathcal{N}$ are not all equally likely to get excited, so this really is a new model, and currently we can not say much about it.

Statistically, it would also be very interesting if we could determine the parameters of our model (the shape of neighbourhood, the probability parameter $p$, the threshold $\theta$) just by looking at the occupied set at some (unknown, large) time. This also looks like a mission impossible at the time.

There are lots of open questions in this theory at the time. As always we find that answering one question opens lots of new ones.

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