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Relationship Between a Restricted Correlated Uniqueness Model and a Direct Product Model for Multitrait-Multimethod Data

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Abstract

This article proves the equivalence between two models which have been suggested for modeling multiplicative method effects in multitrait-multimethod matrices according to the definitions given in Campbell and O'Connell (1967). These are the direct product model (Browne, 1984) and a correlated uniqueness model (Marsh, 1989) with the constraints given by Coenders and Saris (in press). This equivalence is shown to hold for designs with three methods. For designs with more than three methods the constrained correlated uniqueness model is more parsimonious and is equivalent to a direct product model in which method effects can be unidimensionally ranked according to their measurement quality. This equivalence will allow for a widespread use of the correlated uniqueness model, which is a particular class of a confirmatory factor analysis model and hence more easily accessible and understandable by applied researchers. The proof is accompanied by an illustration on real data.

1 Introduction

Multitrait-Multimethod (MTMM) designs (Campbell and Fiske, 1959) consist of multiple measures of a set of factors (*traits*) with the same set of measurement procedures (*methods*). So these designs include $t{\times}m$ measures, that is the number of methods (m) times the number of traits (t).

The differences between methods can be any design characteristic which can be shared by measurements of all traits, such as different response scale lengths or category labels in questionnaires, different data collection procedures, different raters, etc.

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Campbell and Fiske (1959) suggested using MTMM designs for convergent and discriminant validation by directly examining the elements of the correlation matrix among all *txm* measurements, called *MTMM matrix*. This approach was cumbersome and often led to confusion (Schmitt and Stults, 1986) so that from the early seventies MTMM matrices began instead to be analyzed by means of *covariance structure analysis models* (see for instance Bollen, 1989 as a general reference and Schmitt and Stults (1986) for applications on MTMM data). These models are called MTMM models.

Many different MTMM models have been suggested in the literature. Among them are the *correlated uniqueness* (CU) model (Marsh, 1989; Marsh and Bailey, 1991), the *confirmatory factor analysis* (CFA) model for MTMM data (Althauser, Heberlein and Scott, 1971; Alwin, 1974; Werts and Linn, 1970), the *direct product* (DP) model (Browne, 1984, 1985), and the *true score* model for MTMM data (Saris and Andrews, 1991).

The discussion about which of the above models should be preferred has traditionally been linked to a discussion on the behaviour of *method effects*, which determines to a large extent the structure of the MTMM matrix. Campbell and O'Connell (1967) described two such alternative structures linked to two types of method effects: *additive* and *multiplicative*. At that time, the direct examination of correlation coefficients was still common practice so that Campbell and O'Connell's conceptualization was not model-based but referred to the correlations themselves.

It took some time before clear links could be established between the concepts of additive and multiplicative correlation structures and the models used by practitioners. In fact, the terminology used in 1967 by Campbell and O'Connell led to some misunderstanding. Some literature suggested that additive statistical models such as the CFA and CU model were suitable only for analyzing additive MTMM structures (for instance Bagozzi, 1993; Bagozzi and Yi, 1991; Cudeck, 1988; Schmitt and Stults, 1986). The DP model was expressly developed for multiplicative structures and, partly as a result of this misunderstanding, quickly gained popularity.

Kumar and Dillon (1992) pointed out that, in principle, the additive versus multiplicative conceptualization in Campbell and O'Connell (1967) has nothing to do with the additive versus multiplicative formulation of the models. Although these authors, and also Dudgeon (1994), stated under which circumstances the CFA model produces additive or multiplicative structures, their conclusions were not at all straightforward. Browne (1989) also did a great deal of analytical work in order to relate the DP model to a competing additive *components of covariance* model suggested in Wothke (1984).

Coenders and Saris (in press) showed that it was possible to formulate constrained versions of the CU model which produced multiplicative structures in line with Campbell and O'Connell. The relationship between the multiplicative version of the CU model and the DP model is not straightforward and is dealt with in this article, in which it is shown under which circumstances both models are equivalent. This equivalence will allow researchers to use the restricted CU model, which is related to factor analysis models and therefore more accessible and understandable.

2 MTMM matrices and models

2.1 The structure of an MTMM matrix

We next introduce some general comments on the elements of an MTMM matrix and discuss the typical structure of such a matrix. A survey of life satisfaction carried out in Catalonia (Spain) in 1989 will be used for illustration throughout the article. The study considered $t = 3$ domains of life satisfaction (traits):

1. Life as a whole (t1).
2. Financial situation (t2).
3. Social contacts (t3).

Each trait was measured with $m = 3$ response scales ranging from "completely dissatisfied" to "completely satisfied" (methods):

1. 100-point numeric scale (m1).
2. 5-point scale with all-labelled categories (m2).
3. 11-point scale (m3).

Details on the questionnaire and data collection can be found in Batista-Foguet, Coenders and Sureda (1996). The sample size is *406*. The correlation matrix and the standard deviations of the nine measurements are given in Table 1.

Campbell and Fiske (1959) suggested summarizing an MTMM data set by means of the MTMM matrix which arranges the correlation matrix among all measurements ordered by method. The authors referred to the different elements of such an MTMM correlation matrix as follows: *monotrait-heteromethod* correlations involve two measurements of the same trait using different methods; *heterotrait-monomethod* correlations involve two measurements of different traits using the same method; and *heterotrait-heteromethod* correlations involve two measurements of different traits using different methods. The labels of the variables in Table 1 and the presentation of the matrix in blocks following the method should help the reader locate the different types of correlations.

All measures contain random measurement errors. In addition to these errors, the methods used often produce a systematic response error which is called method effect. So, in addition to trait or valid variance, MTMM measurements have two sources of error variance: noise or random error variance and method or invalid variance. Since the second source of error variance is common for all measurements using the same method, the resulting error terms will be correlated.

Table 1: Correlations and standard deviations of nine measurements of life satisfaction

Variable	t1 m1	t2 m1	t3 m1	t1 m2	t2 m2	t3 m2	t1 m3	t2 m3	t3 m3
t1 m1	1.000								
t2 m1	.464	1.000							
t3 m1	.340	.223	1.000						
t1 m2	.574	.241	.330	1.000					
t2 m2	.349	.762	.195	.300	1.000				
t3 m2	.175	.019	.646	.390	.119	1.000			
t1 m3	.639	.346	.309	.630	.326	.281	1.000		
t2 m3	.399	.788	.166	.251	.791	.052	.420	1.000	
t3 m3	.276	.121	.657	.282	.143	.692	.445	.237	1.000
Standard deviation	24.079	24.901	22.986	0.943	1.054	0.937	2.163	2.325	2.102

Random measurement errors tend to attenuate the correlations among observed measurements with respect to the correlations among the trait factors. On the contrary, correlated measurement errors usually increase the correlations among observed measurements in absolute value (at least if trait correlations are positive). In an MTMM correlation matrix, heterotrait-monomethod correlations (which are in the triangular blocks in Table 1) are in general larger in absolute value than heterotrait-heteromethod correlations (which are outside the diagonal of the square blocks in Table 1). In this respect, see for instance Andrews (1984).

Additive and multiplicative MTMM structures differ by the pattern of the aforementioned differences between heteromethod and monomethod correlations. Campbell and O'Connell (1967) suggested plotting monomethod correlations (vertical axis) against heteromethod correlations (horizontal axis) for all pairs of traits. The structure was called additive if, for any given two methods, there was a unit-slope linear relationship between the heterotrait-monomethod correlations involving one of the two methods and the heterotrait-heteromethod correlations. The structure was called multiplicative if the slope of such relationship was larger than one, in other words, if the differences between monomethod and heteromethod correlations were higher when the heteromethod correlations were higher.

MTMM models are useful to provide the researcher with measurement quality estimates (usually in the form of a variance decomposition into trait, error, and method variance) and corrected trait correlations taking the effect of random errors and method effects into account. Among them are the CU and DP models, which are next reviewed.

2.2 Correlated uniqueness (CU) model

The CU model (Kenny, 1976; Marsh, 1989; Marsh and Bailey, 1991) belongs to the family of factor analysis models. The model is specified as:

$$x_{ij} = \lambda_{ij}\,\xi_i + \delta_{ij} \qquad \forall i, j \qquad (1)$$

where x_{ij} is the measurement of trait i with method j, expressed in deviations from the mean; δ_{ij} is the random measurement error plus method effect component for x_{ij} ; ξ_i are the standardized trait factors with correlations $\phi_{ii'}$; and λ_{ij} is the loading of x_{ij} on ξ_i (when standardized and squared it can be interpreted as a measurement quality indicator).

The specification includes the conventional assumption of no correlation between trait factors and error terms:

$$\text{cov}(\delta_{ij}\ \xi_{i'}) = 0 \qquad \forall ij, i' \qquad (2)$$

where i, i',... identify the traits, and j, j',... the methods in all equations in this article. Note that i may be equal to i' and j may be equal to j', unless the opposite is expressly stated.

Covariances among error terms corresponding to pairs of variables measured with the same method (*monomethod error covariances*) constitute unrestricted model parameters which are represented as $\text{cov}(\delta_{ij}\ \delta_{i'j})$. The inclusion of such parameters is a very straightforward manner of accounting for method effects. Covariances among error terms corresponding to pairs of variables measured with two different methods (*heteromethod error covariances*) are constrained to be zero:

$$\text{cov}(\delta_{ij}\ \delta_{i'j'}) = 0 \qquad \text{if } j \neq j' \qquad (3)$$

The restriction in Equation 3 implies that only error covariances among indicators sharing the same method can be explained by the CU model. In some circumstances, some of the methods of measurement are so similar that Equation 3 cannot be expected to hold (de Wit, 1994). For instance, Andrews and Withey (1976, chap. 6) consider six methods for measuring life satisfaction, five based on self-ratings and one on other's ratings. The authors expected correlated measurement errors to occur among all self-rating measures. In such a case the CU model would be misspecified. The literature suggests that such phenomena are quite frequent and high heteromethod error covariances are to be expected when all methods consist in self rating (e.g. Bagozzi, 1993). Fortunately, some literature suggests that, even when heteromethod error covariances are present, the bias of the estimates when fitting the CU models is fairly minor (Marsh and Bailey, 1991; Saris, 1990a; Scherpenzeel, 1995), at least if method effects are low. In any case, models that eliminate the assumption in Equation 3 have been shown to be too heavily parametrized and to lead to problems such as failure to converge, inadmissible estimates, or empirical underidentification (Andrews, 1984; Bagozzi and Yi, 1991; Brannick and Spector, 1990; Kenny and Kashy, 1992; Marsh and Bailey, 1991; Rindskopf, 1984; Saris, 1990b).

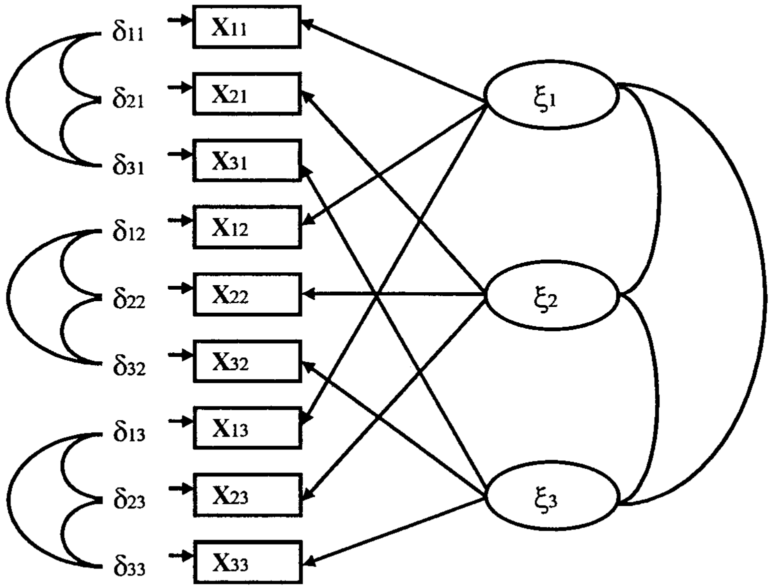

Figure 1: Path diagram of the CU model for three traits and three methods

The path diagram in Figure 1 displays the model for a design with $t = 3$ traits and $m = 3$ methods. The variables measured with the same method appear together in the diagram. Note that each variable is only affected by one trait factor and an error term. Note also the error covariances for the pairs of variables measured with the same method. The CU model can be shown to lead to the following implied variances and covariances:

$$\begin{aligned} &\mathrm{var}(x_{ij}) = \lambda^2_{ij} + \mathrm{var}(\delta_{ij}) && \\ &\mathrm{cov}(x_{ij}\, x_{i'j'}) = \lambda_{ij}\, \phi_{ii'}\, \lambda_{i'j'} && \text{if } i \neq i' \text{ and } j \neq j' \\ &\mathrm{cov}(x_{ij}\, x_{i'j}) = \lambda_{ij}\, \phi_{ii'}\, \lambda_{i'j} + \mathrm{cov}(\delta_{ij}\, \delta_{i'j}) && \text{if } i \neq i' \\ &\mathrm{cov}(x_{ij}\, x_{ij'}) = \lambda_{ij}\, \lambda_{ij'} && \text{if } j \neq j' \end{aligned} \qquad (4)$$

where $\mathrm{var}(x_{ij})$ is a variance, $\mathrm{cov}(x_{ij}\, x_{i'j'})$ is a heterotrait-heteromethod covariance, $\mathrm{cov}(x_{ij}\, x_{i'j})$ is a heterotrait-monomethod covariance and $\mathrm{cov}(x_{ij}\, x_{ij'})$ is a monotrait-heteromethod covariance. The terms $\mathrm{cov}(\delta_{ij}\, \delta_{i'j})$ which are added to the heterotrait-monomethod covariances generally make them larger than the corresponding heterotrait-heteromethod covariances. The equation also decomposes the variance of the x_{ij} variables into trait variance (λ^2_{ij}) and error variance (var(δ_{ij})), the latter including both random error and method variance.

Coenders and Saris (in press) showed that, despite having an additive formulation, the CU model can account for multiplicative method effects if certain constraints are introduced in the error covariance parameters:

$$\text{cov}(\delta_{ij}\ \delta_{i'j}) = c_j\, \phi_{ii'}\, \lambda_{ij}\, \lambda_{i'j} \qquad \forall j,\ i \neq i' \qquad (5)$$

where $c_j \geq 0$ is a constant related to the Method j, $\phi_{ii'}$ is the correlation between Traits i and i' and the λ's are trait loadings. The larger cov(δ_{ij} $\delta_{i'j}$) is, the larger the increase of monomethod correlations or covariances with respect to their heteromethod counterparts is. Thus, the modeling of multiplicative method effects implies that error covariance parameters related to Method j are larger in absolute value when the trait correlation is larger in absolute value. This is precisely the role of the term $\phi_{ii'}$ in Equation 5, which makes error covariances to be *proportional* to the trait correlations. The requirement for c_j to be positive is introduced to accomodate the empirically observed larger size of monomethod correlations with respect to heteromethod correlations and makes error covariances positive if $\phi_{ii'} > 0$ and negative if $\phi_{ii'} < 0$. The λ loadings play the role of scaling constants and make the constrained model *scale invariant* in the sense given by Cudeck (1989). The model can thus be fitted both to covariance and correlation matrices. Moreover, the $\phi_{ii'}$ and c_j parameters are *scale free*, which means that they take the same value regardless of whether a covariance or a correlation matrix is analyzed.

Coenders and Saris (in press) show that the correlation structure implied by the constrained CU model fulfils the definition of multiplicative structure given by Campbell and O'Connell in 1967.

The estimation of the constrained CU model is possible with standard software for structural equation models as long as non linear constraints are permitted. An input file for the LISREL8 program (Jöreskog and Sörbom, 1989, 1993) is given in the Appendix.

A particularly interesting case of the CU model is the *congeneric measurement* (CM) model, one of the most commonly used measurement error models (Jöreskog, 1969, 1971). It is not an MTMM model but its simplicity is such that it has sometimes been used for the analysis of MTMM data, especially if method effects are low (Widaman, 1985).

The specification of the CM model is identical to that of the CU model except for Equation 3, which is reexpressed as:

$$\text{cov}(\delta_{ij}\ \delta_{i'j'}) = 0 \qquad \text{if } i \neq i' \text{ or } j \neq j' \qquad (6)$$

so that both heteromethod and monomethod error covariances are constrained to zero and method effects are thus ignored. Note the absence of the $\mathrm{cov}(\delta_{ij}\ \delta_{i'j'})$ terms from the implied covariances, in comparison to Equation 4:

$$
\begin{aligned}
&\mathrm{var}(x_{ij}) = \lambda^2_{ij} + \mathrm{var}(\delta_{ij}) && \\
&\mathrm{cov}(x_{ij}\ x_{i'j'}) = \lambda_{ij}\ \phi_{ii'}\ \lambda_{i'j'} && \text{if } i \neq i' \text{ and } j \neq j' \\
&\mathrm{cov}(x_{ij}\ x_{i'j}) = \lambda_{ij}\ \phi_{ii'}\ \lambda_{i'j} && \text{if } i \neq i' \\
&\mathrm{cov}(x_{ij}\ x_{ij'}) = \lambda_{ij}\ \lambda_{ij'} && \text{if } j \neq j'
\end{aligned}
\qquad (7)
$$

The constrained CU model is not only related to Campbell and O'Connell's definitions but also to another frequently used models for the analysis of multiplicative MTMM data, namely the DP model, which is next presented. One further section will discuss the equivalence of both models.

2.3 Direct product (DP) model

The DP model has been developed drawing from the work of Swain (1975) with the specific aim of modeling multiplicative method effects. Unlike the previous models, the DP Model (Browne, 1984, 1985, 1989; Cudeck, 1988) is not a factor analysis model. It does not assume the variables to be related through a set of equations but only specifies a certain implied structure for the variances and covariances[3], which is displayed in Equation 8:

$$
\begin{aligned}
&\mathrm{var}(x_{ij}) = (1 + e_{ij})z^2_{ij} && \\
&\mathrm{cov}(x_{ij}\ x_{i'j'}) = z_{ij}\ \pi_{jj'}\ \rho_{ii'}\ z_{i'j'} && \text{if } i \neq i' \text{ and } j \neq j' \\
&\mathrm{cov}(x_{ij}\ x_{i'j}) = z_{ij}\ \rho_{ii'}\ z_{i'j} && \text{if } i \neq i' \\
&\mathrm{cov}(x_{ij}\ x_{ij'}) = z_{ij}\ \pi_{jj'}\ z_{ij'} && \text{if } j \neq j'
\end{aligned}
\qquad (8)
$$

where x_{ij} is the measurement of Trait i with Method j, expressed in deviations from the mean; z_{ij} is a scaling constant for x_{ij}; $\rho_{ii'}$ is the correlation between Traits i and i'; $\pi_{jj'}$ is a method effect indicator[4] (in this model such effects are not given for every

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[3]The DP model can also be viewed as a model for the observed measurements, which specifies them as the *product* of two independent trait and method factors. However, the interpretation of such a multiplicative model for the measurements is problematic and involves rethinking the current measurement theory (Kumar and Dillon, 1992; Marsh and Grayson, 1995). In this section, the DP model will be considered only in terms of its implications for the covariances or correlations. The paper will show that the DP covariance structure is also compatible with an additive model (a restricted CU model) for the observed measurements, whose interpretation is more appealing. The assumptions of the DP model can then be understood to be the same as those of the appropriate restricted CU model.

[4]The model is sometimes interpreted in a symmetric fashion in which the $\pi_{ii'}$ and $\rho_{jj'}$ terms play exactly the same role and are interpreted as correlations (Browne, 1984, 1985; Cudeck, 1988). Traits and methods are viewed as two facets of the MTMM design. Throughout this article it is assumed that

individual method, but for every combination of two of them); and e_{ij} can be understood as a (standardized) random error variance. As shown in the equation, e_{ij} contributes to the variances and not to the covariances and it thus leads to an attenuation of correlations in the same way as a random error component does. A measurement quality indicator can be obtained as $1/(1 + e_{ij})$. An unstandardized random error variance can be obtained as $e_{ij} z^2_{ij}$;

The DP model is scale invariant and the parameters e_{ij}, $\rho_{ii'}$ and $\pi_{jj'}$ are scale free.

The $\rho_{ii'}$, $\pi_{jj'}$ and $\pi_{jj'}$ $\rho_{ii'}$ terms in Equation 8 constitute the most interesting part of the model and are interpreted as correlations between the corresponding x variables disattenuated for the effect of random measurement error. These *disattenuated correlations* take three possible forms:

1. A disattenuated heterotrait-monomethod correlation is equal to the $\rho_{ii'}$ correlation between both traits.
2. A disattenuated heterotrait-heteromethod correlation is equal to the product of the $\rho_{ii'}$ correlation between both traits and the $\pi_{jj'}$ coefficient corresponding to the combination of both methods. The DP model takes method effects into account by means of these $\pi_{jj'}$ coefficients because these coefficients are constrained to be equal to or lower than *1* in absolute value and therefore cause a reduction in the heterotrait-heteromethod correlations with respect to the heterotrait-monomethod correlations. Thus, the lower the $\pi_{jj'}$'s are, the larger the method effects are; if all the $\pi_{jj'}$ coefficients were equal to *1*, no reduction in the correlations would take place, which could be interpreted as the absence of method effects. In fact, it will later be shown that a DP model in which all $\pi_{jj'}$ coefficients are constrained to be equal to 1 is equivalent to the CM model. Note that, in the DP model, method effects reduce heteromethod correlations whereas in the CU model they increase monomethod correlations. Note also that this reduction in the heteromethod correlations is proportional to the trait correlations: it is clear that the difference between the heteromethod and monomethod correlations will be higher when the trait correlation is higher, thus leading to a multiplicative MTMM structure in the sense given by Campbell and O'Connell (1967).
3. A disattenuated monotrait-heteromethod correlation is equal to the corresponding $\pi_{jj'}$ coefficient. This leads to another interpretation of the $\pi_{jj'}$ coefficients which is by no means in conflict with the former one: once random measurement error has been taken into account, a monotrait-heteromethod correlation should be *1* except for the impact of method effects.

In the past, the model could not be estimated with standard software for structural equation modeling unless it was reparametrized as a second order factor

researchers are mainly interested in corrected trait correlations and in measurement quality, which makes the interpretation of $\rho_{ii'}$ as a trait correlation and of $\pi_{jj'}$ as a method effect indicator more attractive.

analysis model (Wothke and Browne, 1990), which was very cumbersome and hard to interpret. As an alternative, Browne's program MUTMUM (Browne, 1985) could be used. Nowadays, the inclusion of non-linear constraints in some standard software for structural equation models makes the estimation easier. A parametrization of the DP model as a first order factor analysis model with non-linear constraints is shown in the appendix, where an input file for the LISREL8 program is also given.

3 Relationship between the CU and DP models

This section will show the conditions under which the constrained CU model specified with Equations 1, 2, 3 and 5 and the DP model are equivalent. The equivalence between the models will be both analytically derived and illustrated on the data in Table 1, which will allow the reader to better relate the parameter values of the equivalent models. Two equivalent models have the same number of free parameters and, for any set of parameters of one model, there exists a set of parameters of the other model which yields an identical implied covariance matrix. For two models to be locally equivalent, equivalence only needs to hold in the regions of the parameter spaces where the parameter values of both models are admissible and both models are empirically identified. Local equivalence is proven by just showing the aforementioned existence of a set of parameters which yields an identical implied covariance matrix. See Luijben (1989) for a more formal definition and examples. Throughout the article we understand equivalence as meaning *local equivalence*, as in Luijben (1989).

3.1 Equivalence between the CM and DP models

Prior to relating the CU and DP models, we will show that the CM model is equivalent to a DP model in which all $\pi_{jj'}$ elements are constrained to *1* so that there are no method effects. We will call this model a *no-method-effect DP model.* This equivalence will shed light on the relationship between the DP model and models of the factor analysis family. This is particularly relevant due to the fact that the DP model has always been presented as belonging to a completely different class of models. Trivially, the CM model is also equivalent to a CU model in which all cov$(\delta_{ij}\ \delta_{i'j})$ error covariance parameters are constrained to zero (i.e. in which there are no method effects either), which increases the convenience of using the CM model as a starting point for the comparison of the CU and DP models.

The no-method-effect DP and CM models have the same number of free parameters: mt z_{ij} or λ_{ij} parameters; $t(t-1)/2$ $\rho_{ii'}$ or $\phi_{ii'}$ parameters; and mt e_{ij} or var(δ_{ij}) parameters. The variances and covariances implied by the no-method-effect DP model can be rewritten from Equation 8 as:

$$\begin{aligned}
\text{var}(x_{ij}) &= (1 + e_{ij})z^2_{ij} \\
\text{cov}(x_{ij}\, x_{i'j'}) &= z_{ij}\, \rho_{ii'}\, z_{i'j'} \\
\text{cov}(x_{ij}\, x_{i'j}) &= z_{ij}\, \rho_{ii'}\, z_{i'j} \\
\text{cov}(x_{ij}\, x_{ij'}) &= z_{ij}\, z_{ij'}
\end{aligned} \tag{9}$$

If we let $\lambda_{ij} = z_{ij}$; $\phi_{ii'} = \rho_{ii'}$; and $\text{var}(\delta_{ij}) = z^2_{ij}\, e_{ij}$, then Equation 7, which contains the variances and covariances implied by the CM model, becomes identical to Equation 9:

$$\begin{aligned}
\text{var}(x_{ij}) &= \lambda^2_{ij} + \text{var}(\delta_{ij}) = z^2_{ij} + z^2_{ij}\; e_{ij} = (1 + e_{ij})z^2_{ij} \\
\text{cov}(x_{ij}\, x_{i'j'}) &= \lambda_{ij}\, \phi_{ii'}\, \lambda_{i'j'} = z_{ij}\, \rho_{ii'}\, z_{i'j'} \\
\text{cov}(x_{ij}\, x_{i'j}) &= \lambda_{ij}\, \phi_{ii'}\, \lambda_{i'j} = z_{ij}\, \rho_{ii'}\, z_{i'j} \\
\text{cov}(x_{ij}\, x_{ij'}) &= \lambda_{ij}\, \lambda_{ij'} = z_{ij}\, z_{ij'}
\end{aligned} \tag{10}$$

Thus, for any set of the no-method-effect DP model parameters there exists a set of CM model parameters which makes the implied variances and covariances of both models the same. So, it has been proven that the CM model and the no-method-edffect DP model are equivalent. This equivalence holds for any values of t and m as long as both models are identified.

3.2 Equivalence between the restricted CU and DP models

We are now in a better position to understand the equivalence between the constrained CU model and the standard DP model. The variances and covariances implied with the constrained CU model are similar to those in Equation 7 except for the addition of the parameters in Equation 5:

$$\begin{aligned}
\text{var}(x_{ij}) &= \lambda^2_{ij} + \text{var}(\delta_{ij}) & & \\
\text{cov}(x_{ij}\, x_{i'j'}) &= \lambda_{ij}\, \phi_{ii'}\, \lambda_{i'j'} & & \text{if } i \neq i' \text{ and } j \neq j' \\
\text{cov}(x_{ij}\, x_{i'j}) &= \lambda_{ij}\, \phi_{ii'}\, \lambda_{i'j}\ \ (1 + c_j\ ) & & \text{if } i \neq i' \\
\text{cov}(x_{ij}\, x_{ij'}) &= \lambda_{ij}\, \lambda_{ij'} & & \text{if } j \neq j'
\end{aligned} \tag{11}$$

These c_j parameters lead to an increase in the monomethod covariances. The unconstrained DP model in Equation 8 also introduces a set of additional $\pi_{jj'}$ parameters with respect to Equation 9. These additional $\pi_{jj'}$ parameters lead to a decrease in the heteromethod covariances. These two different ways of introducing method effects by the CU and DP models will result in the CU λ_{ij} loadings no longer being equal to the z_{ij} parameters. Neither can the $\text{var}(\delta_{ij})$ parameters be directly obtained from the e_{ij} parameters as before. The relationship between the CU and DP models is no longer straightforward and will require certain additional conditions.

A necessary condition for equivalence is that the number of parameters of both models be the same. Both models have tm z_{ij} or λ_{ij} parameters, $t(t-1)/2$ $\rho_{ii'}$ or $\phi_{ii'}$ parameters and tm e_{ij} or var(δ_{ij}) parameters. However, the unconstrained DP model has $m(m-1)/2$ $\pi_{jj'}$ parameters while the constrained CU model has m c_j parameters and these two numbers of parameters coincide only for $m = 3$.

For this reason, we will consider a constrained DP model in which the $\pi_{jj'}$ coefficients can be decomposed into method specific coefficients:

$$\pi_{jj'} = d_j \, d_{j'} \qquad \forall j \neq j' \tag{12}$$

where $0 < d_j \leq 1$ and $0 < d_{j'} \leq 1$ are the coefficients of Method j and Method j'. The lower the value of d_j is, the higher the method effects of Method j are. In the so constrained DP model there exist m d_j parameters, so that the number of parameters of the constrained DP model equals that of the constrained CU model. The implied variances and covariances in Equation 8 must be rewritten as:

$$\begin{array}{ll} \text{var}(x_{ij}) = (1 + e_{ij})z^2_{ij} & \\ \text{cov}(x_{ij}\, x_{i'j'}) = d_j z_{ij}\, \rho_{ii'}\, d_{j'} z_{i'j'} & \text{if } i \neq i' \text{ and } j \neq j' \\ \text{cov}(x_{ij}\, x_{i'j}) = z_{ij}\, \rho_{ii'} z_{i'j} & \text{if } i \neq i' \\ \text{cov}(x_{ij}\, x_{ij'}) = d_j z_{ij}\, d_{j'} z_{ij'} & \text{if } j \neq j' \end{array} \tag{13}$$

We next prove that the constrained CU model with the constraint in Equation 5 and the constrained DP model with the constraint in Equation 12 are equivalent by showing that a set of parameters of the constrained CU model can be obtained from the parameters of the constrained DP model and yields identical implied variances and covariances.

The parameters of the constrained CU model can be expressed from the parameters of the constrained DP models as follows. First, the λ's can be expressed as a function of the d's and the z's as:

$$\lambda_{ij} = d_j z_{ij} \tag{14}$$

so that the λ's will be equal or lower than the z's. The larger z's in the constrained DP model can be understood as a compensation for the reduction in the covariances caused by the $\pi_{jj'}$ terms.

The constrained CU error variances can be expressed as:

$$\text{var}(\delta_{ij}) = z^2_{ij}(e_{ij} + 1 - d^2_j) \tag{15}$$

which shows that they depend on the corresponding z's and e's, and also on a constant linked to the method. Note that these error variances are equal or larger than $z^2_{ij}\, e_{ij}$, which are the error variances that one would compute directly from the constrained DP model parameters. This is so because in the constrained CU model, correlations are only attenuated by random measurement errors, while in the constrained DP model they are further reduced by the $\pi_{jj'}$ parameters. In can even be the case that the e's are negative, which constitutes a non-admissible solution of the constrained DP model, while the $\text{var}(\delta_{ij})$'s are positive.

What is most interesting about the parameter relationships is that they result in trait correlations being numerically the *same* for both models, in spite of the different way in which they take method effects into account:

$$\rho_{ii'} = \phi_{ii'} \tag{16}$$

Finally, the CU c's can be expressed as:

$$c_j = 1/d^2_j - 1 \tag{17}$$

which shows that error covariances will be larger if d is lower. The d's are in general lower when the π's are lower, that is when method effects are higher.

Given these relationships between the parameters of both models, it can be shown that the implied variances and covariances are the same for both models:

$$\begin{aligned}
\text{var}(x_{ij}) &= \lambda^2_{ij} + \text{var}(\delta_{ij}) = d^2_j\ z^2_{ij} + z^2_{ij}\, e_{ij} + z^2_{ij} - z^2_{ij}\, d^2_j = (1 + e_{ij})z^2_{ij} \\
\text{cov}(x_{ij}\, x_{i'j'}) &= \lambda_{ij}\, \phi_{ii'}\, \lambda_{i'j'} = d_j z_{ij}\ \rho_{ii'}\ d_{j'} z_{i'j'} \\
\text{cov}(x_{ij}\, x_{i'j}) &= \lambda_{ij}\, \phi_{ii'}\, \lambda_{i'j}\ \ (1 + c_j\ ) = d_j z_{ij}\ \rho_{ii'}\ d_j z_{i'j} / d^2_j = z_{ij}\, \rho_{ii'} z_{i'j} \\
\text{cov}(x_{ij}\, x_{ij'}) &= \lambda_{ij}\, \lambda_{ij'} = d_j z_{ij}\ d_{j'} z_{ij'}
\end{aligned} \tag{18}$$

It has thus been proven that both models are equivalent.

As regards the generality of this finding, the requirement in Equation 12 will usually hold if $m = 3$ unless the $\pi_{jj'}$ parameters take *very unusual* values (which would lead to negative c_j values which are considered non-admissible). Thus, the *unconstrained* DP model is equivalent to the constrained CU model when $m = 3$. This is very meaningful because designs with three methods are by far the most common in MTMM research. Designs with less than three methods sometimes lead to models which are underidentified or at least to rather unstable estimates. Designs with more than three methods suffer from the high cost connected with profusely repeated measurement.

If $m > 3$, the constrained CU and DP models are more parsimonious than the standard DP model. More particularly, the constrained CU and DP models assume that one single parameter is enough to express the quality of a method. In other words, methods can be arranged in a continuum according to their quality. Thus, a method with a high value of c_j or a low value of d_j leads to monomethod correlations that are substantially higher than the heteromethod correlations regardless of which

Table 2: Equivalence between the DP and constrained CU models. Comparison of parameter estimates when m = 3

Variable	Standard DP model z_{ij}	Standard DP model e_{ij} [a]	Constrained CU model λ_{ij}	Constrained CU model var(δ)
t1 m1	20.70	0.32	17.65	252.01
t2 m1	24.64	-0.03	21.01	146.34
t3 m1	22.05	0.15	18.80	207.94
t1 m2	0.84	0.26	0.73	0.36
t2 m2	1.08	-0.01	0.93	0.27
t3 m2	0.92	1.08	0.80	0.28
t1 m3	2.00	1.14	1.74	1.56
t2 m3	2.38	-0.07	2.06	1.02
t3 m3	2.14	0.00	1.86	1.15

Traits	$\rho_{ii'}$	Methods	$\pi_{jj'}$	Traits	$\phi_{ii'}$	cov($\delta_{ij}\delta_{i'j}$) m1	m2	m3
t1 t2	.39	m1 m2	.74	t1 t2	.39	54.61	0.09	0.47
t1 t3	.48	m1 m3	.74	t1 t3	.48	60.21	0.09	0.52
t2 t3	.26	m2 m3	.75	t2 t3	.26	38.68	0.06	0.33

d_1	d_2	d_3	c_1	c_2	c_3
.85	.87	.87	.38	.33	.33

	$d.f.$	χ^2 stat.	p value
Standard DP model	21	62.90	< .001
Constrained CU model	21	62.90	< .001

[a] The non-admissible negative estimates were not significantly different from the admissibility boundary of 0 and were not constrained to 0 so as to preserve the equivalence between the models. The problem can be attributed to the fact that error variances expressed by the DP parameters are lower than those expressed by the CU parameters.

other method it is combined with. If $m > 3$ the unconstrained DP model offers the possibility that methods be clustered, in such a way that two methods, both with high method effects but with similar behaviours, may produce high mutual heteromethod correlations, or that two methods, both with low method effects, may produce low mutual heteromethod correlations. The presence of such clusters of methods with similar behaviours implies the presence of heteromethod error covariances, which cannot be accounted for by the CU model. If $m > 3$, then a test of the constraints in Equation 12 could reveal the existence of clusters of methods (i.e. heteromethod error covariances). When $m > 3$ and heteromethod error covariances are present, the standard DP model should be preferred over the constrained DP and CU models. When $m > 3$ and heteromethod error covariances are absent, the more parsimonious constrained DP and CU models should be preferred over the standard DP model. When $m = 3$, the presence of heteromethod error covariances cannot either be tested or taken into account with the DP model.

To illustrate the equivalence of the models and the relationships among their parameter values, we separately fitted both of them to the covariance matrix computed from Table 1. Since $m = 3$, no additional constraints need to be introduced to the DP model to ensure equivalence. Table 2 shows the maximum likelihood estimates of both models, obtained with the LISREL8 program using the input files in the appendix. The c_j and d_j parameters are not directly provided by LISREL8 but they were computed by hand from Equations 5 and 12 and included in the table. According to both types of parameters, Method 1 is the one leading to the highest method effects. Allowing for rounding errors, the remaining parameters can be related as specified in Equations 14 to 17. Note the larger values of the z_{ij} terms with respect to the λ_{ij} terms. Note also that, within a method, CU error covariances are larger for pairs of variables measuring traits whose correlation is larger. The trait correlations are the same for both models, as well as the goodness of fit statistic and the degrees of freedom, as it should always be for two models which are equivalent.

4 Discussion

This article has shown that a CU model with constraints leading to multiplicative method effects as defined by Campbell and O'Connell (1967) is equivalent to the standard DP model for designs with *3* methods and to a constrained DP model for designs with more than *3* methods. This constrained DP model assumes that methods can be placed in a continuum according to their effects.

The DP model has always been presented as belonging to a special class of models, with no apparent links to the models of the factor analysis family, outside the very complicated second-order model discussed by Wothke and Browne (1990). We have shown that the DP and constrained CU models, although formally very different, behave in a very similar way, performing the same corrections on trait correlations and relying on similar assumptions. The derived relationship between the DP and constrained CU models allows us to better interpret the meaning of the parameters and assumptions of the DP model and solve some of the frequent misunderstandings which appear in the literature on this subject.

The fact that the $\pi_{jj'}$ parameters have often been called *method correlations* is the source of one of such misunderstanding. In fact, these parameters have sometimes been interpreted as sources of heteromethod error covariances. Quite differently, high values of $\pi_{jj'}$ (i.e. close to one) do not show that heteromethod error covariances are high but that method effects are low; in other words, that the differences between monomethod and heteromethod correlations are low (DP interpretation), or that monomethod error covariances are low (CU interpretation). In the extreme, if all $\pi_{jj'}$

terms are equal to one, then there are no method effects at all and, accordingly, the DP model becomes equivalent to the CM model. As a result of this misunderstanding, some literature (e.g. Bagozzi, 1993) claims that, unlike the CU model, the standard DP model can account for heteromethod error covariances by means of the $\pi_{jj'}$ parameters. This is indeed so if $m > 3$, but not otherwise.

The claim made by Campbell and O'Connell (1982) that, under some circumstances, monomethod correlations should be more realistic than heteromethod correlations, together with the fact that the DP model trait correlations are related to monomethod correlations, could lead to another misunderstanding: it could be suggested that the trait correlation estimates obtained from the DP model can be more appropriate than those obtained from the CU model. We have shown that the $\rho_{ii'}$ parameters of the constrained DP model equal the $\phi_{ii'}$ parameters of the CU model, so that they both can be interpreted as trait factor correlations which are corrected for both random errors and method effects in exactly the same way.

Researchers willing to model multiplicative method effects can then use the constrained CU model instead of the standard DP model. The CU model offers the advantage of being more comparable to standard models. Note that the CU model may be used also if $m > 3$. In this case, it is only equivalent to the constrained DP model but it continues to model multiplicative method effects in the sense given by Campbell and O'Connell (1967). As far as we know, the CU model has never been used to deal with multiplicative MTMM structures, probably because the additive formulation of the model did not suggest this possibility. Many a researcher who finds the DP model hard to interpret can benefit from adopting the CU family of models instead.

A drawback of the CU model is that it assumes that heteromethod error covariances are zero but models which relax this assumption often lead to problems as was mentioned. The standard DP model can only relax this assumption when $m > 3$. In our opinion, this drawback is more than compensated for by the CU model's lack of practical problems: Marsh (1989) and Marsh and Bailey (1991) report that the CU model rarely leads to problems of empirical underidentification, failure to converge or inadmissible estimates. Moreover these authors show that, even in the cases in which the CU model is misspecified due to the presence of heteromethod error covariances, the CU estimates are closer to the population parameter values than the estimates of aslternative correctly specified but highly unstable models. This is so because the lower sampling variability of the CU estimates outweighs the bias arising from the violation of the model's assumptions.

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Appendix

Parametrization of the constrained CU model

The parametrization we suggest involves the usual specification of a CU model with the appropriate non-linear constraints in the non-zero error covariances. The constraints are introduced in the following way: m cov(δ_{1j} δ_{2j}) error covariances concerning Traits 1 and 2 are taken as free parameters and the remaining cov(δ_{ij} $\delta_{i'j}$) error covariances with $i \geq 1$ and $i' > 2$ are expressed as a function of cov(δ_{1j} δ_{2j}):

The parametrization we suggest involves the usual specification of a CU model with the appropriate non-linear constraints in the non-zero error covariances. The constraints are introduced in the following way: m cov(δ_{1j} δ_{2j}) error covariances concerning Traits 1 and 2 are taken as free parameters and the remaining cov(δ_{ij} $\delta_{i'j}$) error covariances with $i \geq 1$ and $i' > 2$ are expressed as a function of cov(δ_{1j} δ_{2j}):

$$\text{cov}(\delta_{ij}\,\delta_{i'j}) = \text{cov}(\delta_{1j}\,\delta_{2j})\,\frac{\phi_{ii'}\lambda_{ij}\lambda_{i'j}}{\phi_{12}\lambda_{1j}\lambda_{2j}} \qquad \forall j,\ i \geq 1,\ i' > 2$$

A LISREL8 input file for $t = 3$ and $m = 3$ is provided below. Note that in this file subindexes refer to the row and column of a matrix of parameters, and not to the trait and method.

```
LISREL8 INPUT FILE FOR CONSTRAINED CU  MODEL (3 TRAITS X 3 METHODS)
! NO= REQUIRES THE SAMPLE SIZE
DA NI=9 NO=  MA=CM
LA
'tl ml' 't2 ml' 't3 ml' 'tl m2' 't2 m2' 't3 m2' 'tl m3' 't2 m3' 't3 m3'
! CM= REQUIRES A FILENAME FOR A LOWER TRIANGULAR COVARIANCE MATRIX
! VARIABLES MEASURED WITH THE SAME METHOD MUST APPEAR TOGETHER IN THE MATRIX
CM=
MO NX=9 NK=3 LX=FU,FI PH=ST TD=SY,FI
LK
'T1' 'T2' 'T3'
FR LX 1 1 LX 2 2 LX 3 3 LX 4 1 LX 5 2 LX 6 3 LX 7 1 LX 8 2 LX 9 3
FR TD 1 1 TD 2 2 TD 3 3 TD 4 4 TD 5 5 TD 6 6 TD 7 7 TD 8 8 TD 9 9
FR TD 2 1 TD 3 1 TD 3 2 TD 5 4 TD 6 4 TD 6 5 TD 8 7 TD 9 7 TD 9 8
! MULTIPLICATIVE NON-LINEAR CONSTRAINTS ACCORDING TO DEFINITION M2
CO TD(3,1)=TD(2,1)*PH(3,1)*LX(3,3)*PH(2,1)**-1.0*LX(2,2)**-1.0
CO TD(3,2)=TD(2,1)*PH(3,2)*LX(3,3)*PH(2,1)**-1.0*LX(1,1)**-1.0
CO TD(6,4)=TD(5,4)*PH(3,1)*LX(6,3)*PH(2,1)**-1.0*LX(5,2)**-1.0
CO TD(6,5)=TD(5,4)*PH(3,2)*LX(6,3)*PH(2,1)**-1.0*LX(4,1)**-1.0
CO TD(9,7)=TD(8,7)*PH(3,1)*LX(9,3)*PH(2,1)**-1.0*LX(8,2)**-1.0
CO TD(9,8)=TD(8,7)*PH(3,2)*LX(9,3)*PH(2,1)**-1.0*LX(7,1)**-1.0
OU ME=ML
```

Parametrization of the DP model

The parametrization we suggest is that of a factor analysis model with constrained factor covariance matrix. In this parametrization there are *mt* factors and each of the *mt* x_{ij} observed variables loads only on one ξ_{ij} factor and has a zero error variance:

$$x_{ij} = \lambda_{ij}\,\xi_{ij}$$

The factor covariance matrix contains the terms $1 + e_{ij}$ in the diagonal and the disattenuated correlations outside the diagonal. The values outside the diagonal must then be constrained to appropriate products among a set of free $\pi_{jj'}$ and $\rho_{ii'}$ parameters. The λ_{ij} factor loadings play the role of the z_{ij} terms.

Below are the implied variances and covariances of this parametrization and of the DP model:

$$\mathrm{var}(x_{ij}) = \mathrm{var}(\xi_{ij})\lambda^2_{ij} = (1 + e_{ij})z^2_{ij}$$
$$\mathrm{cov}(x_{ij}\,x_{i'j'}) = \mathrm{cov}(\xi_{ij}\,\xi_{i'j'})\lambda_{ij}\,\lambda_{i'j'} = \pi_{jj'}\rho_{ii'}z_{ij}z_{i'j'}$$
$$\mathrm{cov}(x_{ij}\,x_{i'j}) = \mathrm{cov}(\xi_{ij}\,\xi_{i'j})\lambda_{ij}\,\lambda_{i'j} = \rho_{ii'}z_{ij}z_{i'j}$$
$$\mathrm{cov}(x_{ij}\,x_{ij'}) = \mathrm{cov}(\xi_{ij}\,\xi_{ij'})\lambda_{ij}\,\lambda_{ij'} = \pi_{jj'}z_{ij}z_{ij'}$$

A LISREL8 input file for $t = 3$ and $m = 3$ is provided below.

```
LISREL8 INPUT FILE FOR DP MODEL (3 TRAITS X 3 METHODS)
! PARAMETRIZED AS A FA MODEL WITH 9 FACTORS WITH NON-LINEAR CONSTRAINTS
! THE OFF-DIAGONAL PHI MATRIX IS THE 9X9 DISATTENUATED CORRELATION MATRIX
! THE DIAGONAL OF PHI CONTAINS 1+e(ij) AND IS UNCONSTRAINED
! LX CONTAINS THE SCALING FACTORS z(ij)
! NO= REQUIRES THE SAMPLE SIZE
DA NI=9 NO=    MA=CM
LA
't1 m1' 't2 m1' 't3 m1' 't1 m2' 't2 m2' 't3 m2' 't1 m3' 't2 m3' 't3 m3'
! CM= REQUIRES A FILENAME FOR A LOWER TRIANGULAR COVARIANCE MATRIX
! VARIABLES MEASURED WITH THE SAME METHOD MUST APPEAR TOGETHER IN THE MATRIX
CM=
MO NX=9 NK=9 TD=ZE LX=DI,FR PH=SY,FR
LK
'T1M1' 'T2M1' 'T3M1' 'T1M2' 'T2M2' 'T3M2' 'T1M3' 'T2M3' 'T3M3'
! RESTRICTIONS TO THE STRUCTURE OF THE DISATTENUATED CORRELATION MATRIX
! DISATTENUATED HETEROTRAIT-MONOMETHOD CORREL. rho(ii') EQUAL FOR ALL METHODS
EQ PH 2 1 PH 5 4 PH 8 7
EQ PH 3 1 PH 6 4 PH 9 7
EQ PH 3 2 PH 6 5 PH 9 8
! DISATTENUATED MONOTRAIT-HETEROMETHOD CORREL. pi(jj') EQUAL FOR ALL TRAITS
EQ PH 4 1  PH 5 2  PH 6 3
EQ PH 7 1  PH 8 2  PH 9 3
EQ PH 7 4  PH 8 5  PH 9 6
! 18 DISATTENUATED HETEROTRAIT-HETEROMETHOD CORRELATIONS
! CONSTRAINED TO THE PRODUCT OF ONE rho(ii') TERM AND ONE pi(jj') TERM
CO PH(5,1)=PH(2,1)*PH(4,1)
CO PH(6,1)=PH(3,1)*PH(4,1)
CO PH(6,2)=PH(3,2)*PH(4,1)
CO PH(4,2)=PH(2,1)*PH(4,1)
CO PH(4,3)=PH(3,1)*PH(4,1)
CO PH(5,3)=PH(3,2)*PH(4,1)
CO PH(8,1)=PH(2,1)*PH(7,1)
CO PH(9,1)=PH(3,1)*PH(7,1)
CO PH(9,2)=PH(3,2)*PH(7,1)
CO PH(7,2)=PH(2,1)*PH(7,1)
CO PH(7,3)=PH(3,1)*PH(7,1)
CO PH(8,3)=PH(3,2)*PH(7,1)
CO PH(8,4)=PH(2,1)*PH(7,4)
CO PH(9,4)=PH(3,1)*PH(7,4)
CO PH(9,5)=PH(3,2)*PH(7,4)
CO PH(7,5)=PH(2,1)*PH(7,4)
CO PH(7,6)=PH(3,1)*PH(7,4)
CO PH(8,6)=PH(3,2)*PH(7,4)
! STARTING VALUES FOR THE ESTIMATION
! WE RECOMMEND GIVING THE LX PARAMETERS VALUES CLOSE TO THE STANDARD
DEVIATIONS
ST  .5 PH 2 1 PH 3 1 PH 3 2
ST  .9 PH 4 1 PH 7 1 PH 7 4
ST  .9 LX 1 1 LX 2 2 LX 3 3 LX 4 4 LX 5 5 LX 6 6 LX 7 7 LX 8 8 LX 9 9
ST 1.3 PH 1 1 PH 2 2 PH 3 3 PH 4 4 PH 5 5 PH 6 6 PH 7 7 PH 8 8 PH 9 9
OU NS ME=ML
```