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Bézier Curves: Simple Smoothers of Noisy Data

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Abstract

In many instances only the approximation curve to experimental or field data is of interest. Application of Bernstein-Bézier polynomials for smoothing of noisy data is shown. Bézier curves provide a simple yet efficient way to approximate (or smooth) arbitrarily spaced vector valued functions without explicit model specification. To evaluate the efficiency of Bézier curves some simulated cases are presented.

Keywords: Bernstein-Bézier polynomials; Smoothing; Noisy data; Simulation; Statistics.

1 Introduction

Smoothing of noisy data is a common problem in data analysis. The ability of human brains to extract the signal from the data, even from substantial noise, is amazing. But we get into troubles when we want to get rid of noise by calculation. If one understands the underlying process, the modelling approach is used. It is not an easy task especially in nonlinear cases. One has to estimate model parameters and to decide about the complexity of the modelling function.

If model parameters have an explanation as process parameters and the estimates are used for comparison of different processes, then it is worthwhile to estimate them. But this is not always the case. Sometimes we want only to drop the noise and plot what we perceive as the signal, hidden in the noise. In such cases hand-drawn curves would be sufficient. It seems, that Bézier curves are a sort of formalized hand-drawn curves.

Approximation with Bernstein polynomials, known from general approximation theory (Isaacson 1966), was recommended by Schoenberg (1959) : "The Bernstein polynomials should be used whenever we need polynomial approximation which does not oscillate more often about any straight line than the function to be approximated." He showed that Bernstein approximation is always at least as "smooth" as the primitive function f where "smooth" refers to the number of undulations and the total variation of f. The main disadvantage of Bernstain approximation

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is slow convergence to f as the degree of polynomial is increased. Davis (1963) suggested that Bernstein polynomials would "perhaps ... find application when the properties of the approximation in the large are of more importance than closeness of fit". One such field of applications, referred by Forrest (1972) as ***computational geometry*** is the design of smooth free-form curves and surfaces (Gordon 1974). In the field of design for the automobile, aircraft and shipbuilding industries modified Bernstein polynomials, known as Bézier curves, are widely used due to P. Bézier who introduced them to automobile design. Bézier curves are used in modern computer design of typefaces for printing (Knuth 1986).

In present article we shall try to show an application of Bézier curves to data smoothing problem.

2 Bernstein polynomials

Bernstein polynomial of degree n associated to an arbitrary function $f : [0,1] \to R$ is defined as

$$B_n[f](t) = \sum_{k=0}^{n} b_{n,k}(t) f(\frac{k}{n}) \tag{1}$$

where the weighting functions $b_{n,k}$ are the discrete binomial probability density functions for fixed probability t

$$b_{n,k}(t) = \binom{n}{k} t^k (1-t)^{n-k}, \quad k = 0, 1, \ldots, n. \tag{2}$$

Regarded as an operator with argument f, B_n is linear, $B_n[\alpha f + \beta g] = \alpha B_n[f] + \beta B_n[g]$ for any real numbers α and β. Bernstein polynomials have variation diminishing and convexity preserving properties (Schoenberg 1959). An example of Bernstein polynomials is shown in Figure 1.

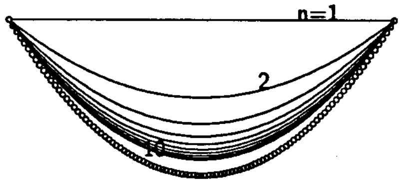

Figure 1: Bernstein polynomials of degrees $n = 1, 2, \ldots, 10$ (lines) for $f(x) = x^2$ (circles).

3 Bézier curves

Bernstein polynomials were modified by P. Bézier for description of vector valued functions : curves $f : [0,1] \to R^2$ or surfaces $f : [0,1]^2 \to R^3$.

Definition 1 *Let $P_k (k = 0, 1, \ldots, n)$ be $n + 1$ ordered points in R^m and consider the polygon formed by joining successive points. The* **Bézier curve** *associated with this polygon is the vector valued Bernstein polynomial $B_n[P_0, P_1, \ldots, P_n]$ given by*

$$B_n[P_0, P_1, \ldots, P_n] = \sum_{k=0}^{n} b_{n,k}(t) P_k, \tag{3}$$

where $b_{n,k}(t)$ are the binomial probability density functions (2).

In the case of a planar curve $P_k(x_k, y_k) \in R^2$ one simply treats each coordinate x and y independently obtaining two equations of (1).

$$B_n[x](t) = \sum_{k=0}^{n} b_{n,k}(t) x_k \tag{4}$$

and

$$B_n[y](t) = \sum_{k=0}^{n} b_{n,k}(t) y_k. \tag{5}$$

to illustrate the relation between polygon $P_k (k = 0, 1, \ldots n)$ and the associated Bézier curve, Figure 2 shows two plane curves.

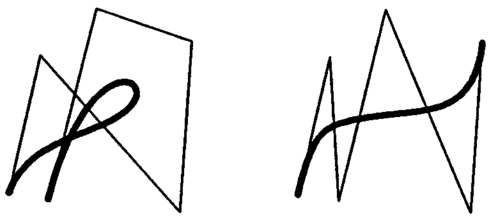

Figure 2: Two polygons (thin line) with associated Bézier curves (thick line)

The ordering of data points influences the shape of the associated Bézier curve. In the example from Figure 2, the relative positions of points on the polygons are the same but their order is not. Note also that the Bézier curve passes through the first and last data point with the first and last polygon segment being its tangents.

4 Bézier curves and smoothing of noisy data

Bézier curves were applied to the problem of noise reduction in noisy set of data: Let $x_0 \leq x_1 \leq \ldots \leq x_n$ be a set of ordered arbitrarily spaced points on a finite interval and let $y_0, y_1, \ldots, y_n$ be a corresponding set of noisy observations given by

$$y_k = f(x_k) + \varepsilon_k, \quad (k = 0, 1, \ldots, n) \tag{6}$$

where f is smooth, but unknown, function of x and the ε_k are random errors with $E(\varepsilon) = 0$ and $E(\varepsilon^2) = \sigma^2$ for some unknown σ^2 (E denotes expectation).

One wishes to find a smooth approximation (or regression curve) for the set of points (x_k, y_k).

As "Bernstein approximation is always at least as smooth as the primitive function" (Gordon 1974, p. 296), one can expect to gain such smooth approximation by finding the Bézier curve (4),(5) for the set of points (x_k, y_k), $(k = 0, 1, \ldots, n)$ and approximate them by the corresponding set of points $(B_n[x](t), B_n[y](t))$, $t \in [0, 1]$. From linearity of B_n and definition of y (6) we have

$$B_n[y](t) = B_n[f(x)](t) + B_n[\varepsilon](t), \quad t \in [0, 1].$$

As one can expect $B_n[f(x)]$ to be close to f, goodness of approximation depends on variation of $B_n[\varepsilon]$. From general properties of Bernstein polynomials we can expect, that variance of the error term ε is reduced *i.e.* $E(B_n[\varepsilon^2]) < E(\varepsilon^2)$(see Figure 3).

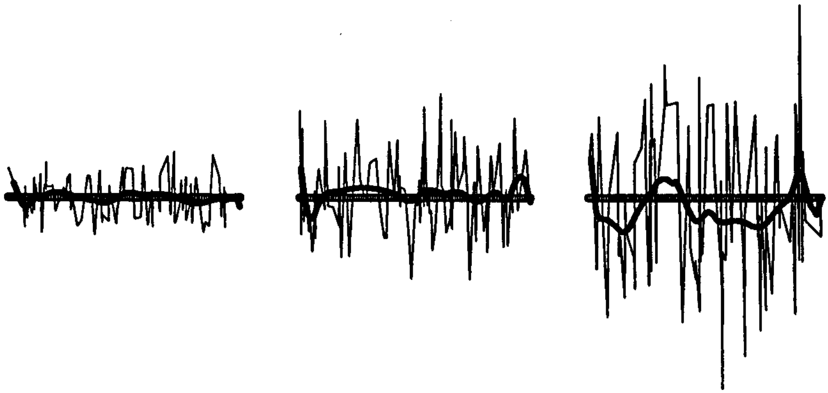

Figure 3: Bézier curve (thick line) for the error term ε (thin line) ($n = 100, \sigma^2 = 20, 40, 80$). Notice the susceptibility of approximation to change in variation. Small circles represent mean value of the error term.

Bézier curves are mean value preserving and variance diminishing (Blejec 1992). (Mean value preserving property is very important in design applications, since the center of gravity is the same for the control polygon and the corresponding Bézier curve/surface.) The mentioned properties enable us to use the proportion of explained variance $r_B^2 = Var(B_n[y])/Var(y)$ (Var stands for variance) as the measure of goodness of approximation of f from (6). However, it is not expected for r_B^2 to be close to 1, if variance of the error term is large. In the best case, r_B^2 is comparable to determination coefficient r_f^2 for f from (6), if the later is known.

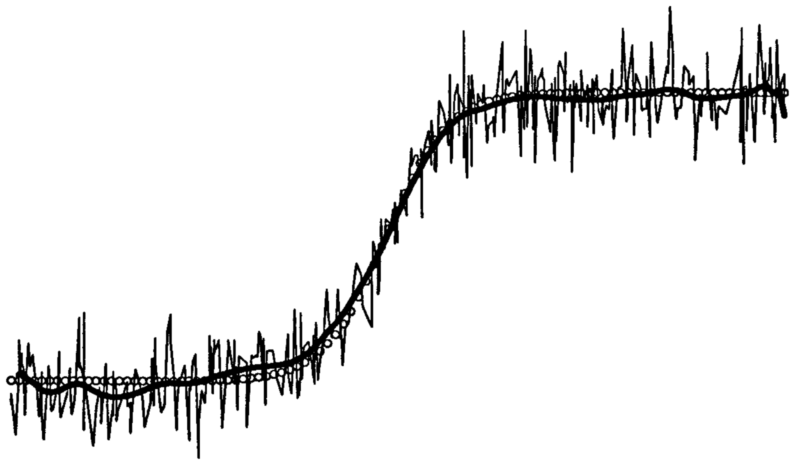

Figure 4: Logistic function (small circles) with added error term (thin line) and associated Bézier curve (thick line) (n=400).

5 Simulation

In order to test the efficiency of Bézier curves for data smoothing, several simulations with known function f and known variance of ε were run. Some results are plotted on Figures 3 to 8.

A computer program for generation of sequences of points $[(x_k, y_k), k = 0, 1, \ldots, n]$, according to (6) was prepared. Bézier curves were calculated from (4) , (5) and plotted against the original data. For a large number of points ($n > 200$), Normal approximation of Binomial probabilities was used. Poisson approximation was used instead of Normal for low ($t < 0.1$) and high ($t > 0.9$) values of t. From the definition of Bézier curve ((3) and (4)) it is clear that data may be arbitrarily spaced on x. Different numbers of data points were used in simulations. Bézier curves were efficient regardless of the number of data points in original data. However, when the variance of the error term is large, it is desirable not to have too few measurements. The calculation scheme is efficient enough to be used with large datasets (several thousand points).

Even if equations (4) and (5) indicate, that all data points contribute to the calculated points on the Bézier curve, it is not the case for large data sets since $b_{n,k}(t)$ vanish while k/n diverge from t. The distribution of weights $b_{n,k}(t)$ is different for each t, being highly asymmetric at the endpoints of the data sequence. We can treat calculation scheme for Bézier curve as a moving average with adaptive weights $b_{n,k}$ for each t. No data points have to be abandoned, and we are able to calculate the approximation from the first data point to the last one. As a disadvantage we may count, that Bézier curve starts at the first and ends at the last data point. But in experiments, the beginning and end of a set of measurements are seldom of interest.

6 Conclusions

In our experiments, the residual variance $Var(y - B_n[y])$ was always close to known variance of the error term ε and r_B^2 was close to r_f^2. From this we conclude, that Bézier curves are good and simple approximations for the true regression curves f.

Due to slow convergence of Bernstein polynomials, an underestimate of curvature was found if original function was highly curved. But this slow convergence ensures the averaging property of Bézier curves.

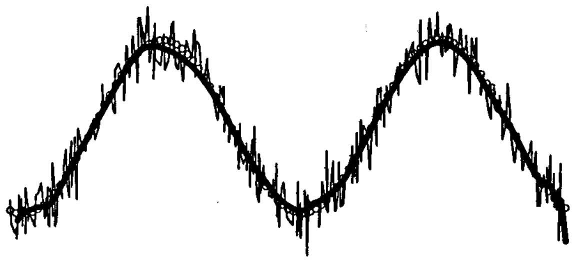

Figure 5: Sine function (small circles) with added error term (thin line) and associated Bézier curve (thick line) ($n = 400$).

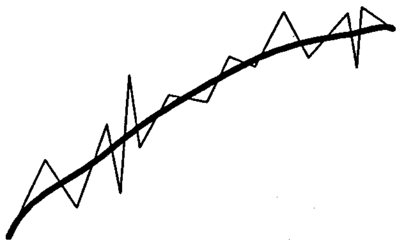

Figure 6: Growth curve (n=17).

We found interesting results in applying Bézier curves to some strange situations that are very common in practice. One such simulated function is shown on Figure 7.

In spite of some inefficiencies we found Bézier curves quite satisfactory in many applications.

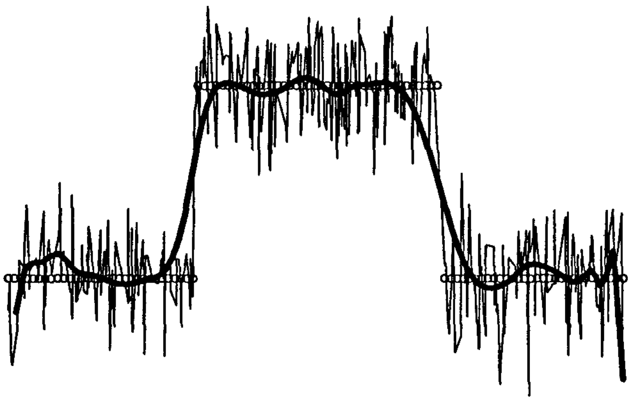

Figure 7: Step function (small circles) with added error term (thin line) and associated Bézier curve (thick line) (n=400).

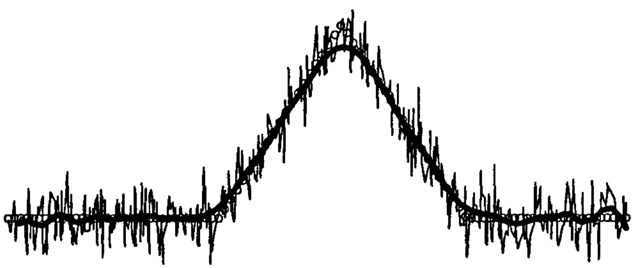

Figure 8: Spike function (small circles) with added error term (thin line) and associated Bézier curve (thick line) (n=500).

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