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Graph Distance in Multicriteria Decision Making Context

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Abstract

In this article we introduce a dissimilarity measure on a certain class of oriented graphs arising in the context of group decision making. Group consensus is done using Potential Method (PM) developed by the author in 2002.

Possible applications of the method are discovering hidden conflicts in society and testing a homogeneity of micro-social structures, like sport teams. Here we use it to find the outliers in the group of students whose task was to compare their lecturers in regard to a certain number of criteria.

1 Introduction

1.1 A brief description of Potential Method

For simplicity, let us suppose that one decision maker makes pairwise comparisons of alternatives, the set of alternatives being denoted by V, using a single criterion.

A pair $\alpha = (u, v) \in V \times V$ is declared to be an arc of a directed graph if v is more preferred than u with weight of that preference a non-negative number. An uncompared pair is not adjacent in the graph. The set of all arcs is denoted by $\mathcal{A}$. A function $F : \mathcal{A} \to \mathbb{R}$ which assigns to each arc $\alpha \in \mathcal{A}$ it's weight is called a *preference flow*. The flow component $F(\alpha)$ is usually taken from a given interval; here we use interval $[0, 4]$ of real numbers. Evidently, preference flow is always non-negative and can be represented as an $m \times 1$ matrix over $[0, 4]$. Oriented graph $(V, \mathcal{A})$ is called *preference graph*. Preference graph is *complete* if each pair of alternatives is compared i.e. if for each pair (i, j) of vertices $(i, j) \in \mathcal{A}$ or $(j, i) \in \mathcal{A}$. For a given flow F on the preference graph $(V, \mathcal{A})$ and $\alpha \in \mathcal{A}$ we use a convention $F(-\alpha) := -F(\alpha)$.

Let us denote by n and m the cardinality of V and $\mathcal{A}$ respectively. Incidence matrix of the preference graph is denoted by B and it is $m \times n$ matrix defined by

$$B_{\alpha,i} = \begin{cases} -1, & \text{if } \alpha \text{ leaves } i \\ 1, & \text{if } \alpha \text{ enters } i \\ 0, & \text{otherwise.} \end{cases}$$

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Let F be a given preference flow, and B incidence matrix of the graph. Let us assume, for simplicity, that the graph is weakly connected (connected in the sequel). The system

$$B^\tau BX = B^\tau F, \quad \sum_{i=1}^{m} X_i = 0 \tag{1.1}$$

has a unique solution. Namely, the null-space of B is one-dimensional and generated by $(1, 1, \dots, 1)$ and the set of all solutions of equation $B^\tau BX = B^\tau F$ is one-dimensional affine space. Each solution of that equation represents 'utility' on the sets of alternatives that should be invariant on adding a constant to each vertex. This is a reason for the second condition in (1.1). One can think of the first equation in (1.1) as the normal equation associated to $BX = F$. The unique solution of (1.1) is called *normal integral* of F. Sometimes, a function $X : V \to \mathbb{R}$ is called potential and this is the reason for name 'Potential Method'. If the graph is not connected, the normal integral is unique on each connected component of the graph. Another name for X is *utility function*, widely used in the literature dealing with decision making. The potential difference BX of normal integral is the best approximation of F by column space of incidence matrix.

To obtain a ranking, after having X, the following formula can be used

$$w = \frac{a^X}{\|a^X\|_1}, \quad a > 0$$

where exponent function of X is defined componentwise, i.e. $(a^X)_i = a^{X_i}$, and $\|\cdot\|_1$ is l_1 norm, usually called the Manhattan or taxi norm. Parameter a can be arbitrary but experience suggests to use value $a = 2$.

Measure of inconsistency is defined as

$$\mathrm{Inc}(F) = \frac{\|F - BX\|_2}{\|BX\|_2},$$

where $\|\cdot\|_2$ denotes 2-norm and $\beta = \arctan(\mathrm{Inc}(F))$ is *angle of inconsistency*. Ranking is considered 'acceptable' if β is less than 12 degrees. The last statement should not be taken for granted, as there is no serious argument to support it due to the freshness of the method.

1.2 Consensus flow

If more than one criterion is present, each criterion C_i generates its own graph $(V, \mathcal{A}_i)$ and its own flow F_i. Let us denote the weight of the i-th criterion by w_i, where $\sum_i w_i = 1$. We are going to describe a procedure of making a *consensus graph* $(V, \mathcal{A})$ and *consensus flow* F for the group of all criteria.

First, for a given pair $\alpha = (u, v)$ we calculate

$$F_\alpha := \sum_{\substack{i=1 \\ \pm\alpha \in \mathcal{A}_i}}^{k} w_i F_i(\alpha)$$

where the term $w_i F_i(\alpha)$ contributes if and only if $\pm\alpha \in \mathcal{A}_i$ i.e. if and only if $F_i(\alpha)$ or $F_i(-\alpha)$ is defined. If this sum is non-negative, then we put α in the set of arcs $\mathcal{A}$ and $F(\alpha) := F_\alpha$. Otherwise, we define $-\alpha = (v, u)$ as an arc in $\mathcal{A}$ and $F(-\alpha) := -F_\alpha$. The flow F becomes a non-negative flow that is called *consensus flow*. It can happen that consensus graph has a cycle. Anyway, normal integral of F exists and it is unique. The presence of cycles can only generate greater inconsistency $\mathrm{Inc}(F)$.

Potential Method satisfies the Row Dominance Property, the Absolute Dominance Property and the axiom of Positive Association of Social and Individual Values. The proof can be found in L. Čaklović (2002).

The classical approach for aggregation of individual preferences into a social preference, as described in Barthélémy (1989), is Kemeny's median (Kemeny, 1959) which to the group profile $\pi(\rho_1, \ldots, \rho_n)$ of linear orders assigns the set of all linear orders ρ such that the sum

$$\sum_{i=1}^{n} \delta(\rho, \rho_i)$$

is minimal. Here $\delta(\rho_i, \rho_j) = |\rho_i \cup \rho_j| - |\rho_i \cap \rho_j|$ denotes the symmetric difference distance between relations.

The social preference obtained by Kemeny's median and the one described here, obtained by Potential Method, are related in the same way as the median and arithmetic mean in elementary statistic.

2 Motivation for graph distance

A profile of a group $\mathcal{G}$ is a $|\mathcal{G}|$-tuple $\pi = (F_1, \ldots, F_{|\mathcal{G}|})$ of individual preference flows.

For the moment, let us suppose that each individual flow is obtained from a linear order ρ_i on the set of vertices, i.e. it's graph is a transitive tournament and $F_i(\alpha) = 1$ for each $\alpha \in \rho_i$. In that case, symmetric difference distance is the l_1-norm $\|F_i - F_j\|_1$. In general case, the flows F_i and F_j belong to different vector spaces, because the number of arcs in preference graphs that corresponds to F_i and F_j are not equal, and $F_i - F_j$ makes no sense. This was our motivation to define a distance between incomplete oriented graphs as an l_1-norm of the difference of their completion, completion being extended by zero on the complement arcs in complete graph. From the point of view of decision maker this is a 'wrong' definition because a value of zero of the flow component $F(\alpha)$, $\alpha = (u, v)$ means that u and v are compared and they are considered equally preferable, which is NOT the case in incomplete preference graph. The following definition seems to be a better one.

Definition 1 *Let $(V, \mathcal{A}_1)$ and $(V, \mathcal{A}_2)$ be two oriented graphs with given preference flows $F_i : \mathcal{A}_i \longrightarrow \mathbb{R}_+$. Let us define*

$$\delta(F_1, F_2) := \|X_1 - X_2\| \tag{2.1}$$

where X_i is the normal integral of F_i and $\|\cdot\|$ some norm on $\mathbb{R}^n$, using the euclidian l_2 norm.

Generally speaking, two decision makers can define two different preference flows on the set of alternatives. If they induce the same value function they are considered equivalent. In this context equation (2.1) gives a good dissimilarity measure on the set of preference flows over the same set of alternatives.

2.1 Flow completion and flow complement

Equation (1.1) provides a way to define a completion of an incomplete flow F. Moreover, we shall prove that the dissimilarity measure defined by (2.1) can be considered as a metric on the quotient space of the space of complete flows with Kirchoff's flows as the kernel. Recall that the space of Kirchoff's flows is defined as the kernel of the transpose of the incidence matrix.

Definition 2 *For a given graph $(V,\mathcal{A})$ let us denote by $\mathcal{A}^c$ a set of arcs such that $(V,\mathcal{A}\cup\mathcal{A}^c)$ is an oriented complete graph. The graph $(V,\mathcal{A}\cup\mathcal{A}^c)$ is a* completion *of $(V,\mathcal{A})$ and $\mathcal{A}^c$ is a* complement *of $\mathcal{A}$.*

A complement *F^c of F with respect with respect to $\mathcal{A}\cup\mathcal{A}^c$, is the flow defined by*

$$F^c(\alpha) = X(b) - X(a), \quad \alpha = (a,b) \in \mathcal{A}^c \tag{2.2}$$

where X denotes the normal integral of F. Evidently, completion and complement are not unique.
A flow

$$\hat{F}(\alpha) = \begin{cases} F(\alpha), & \alpha \in \mathcal{A} \\ F^c(\alpha), & \alpha \in \mathcal{A}^c \end{cases}$$

is called a completion *of F with respect to $\mathcal{A}\cup\mathcal{A}^c$.*

Proposition 3 *Assume that the flows $F_i : \mathcal{A}_i \to \mathbb{R},\ i = 1,2$, have the same normal integral. Then, there exists a complete oriented graph $(V,\mathcal{A})$ and completions $\hat{F}_i,\ i = 1,2$, with respect to $\mathcal{A}$, such that*

$$B^\tau(\hat{F}_1 - \hat{F}_2) = 0.$$

Proof:
By assumption

$$B_i^\tau B_i X = B_i^\tau F, \quad i = 1,2 \tag{2.3}$$

where B_i denotes the incidence matrix of $(V,\mathcal{A}_i)$. Changing the sign of some components of F_i and changing the orientation of the corresponding arcs in $\mathcal{A}_i$ we can obtain that $\mathcal{A}_1 \cap \mathcal{A}_2$ has the maximum possible cardinality. Equations (2.3) remain unchanged after performing these transformation because $B^\tau B$ and $B^\tau F$ are invariant on them.

Let $(\mathcal{A}_1\cup\mathcal{A}_2)^c$ denote any complement of the union $\mathcal{A}_1\cup\mathcal{A}_2$ in the complete graph $(V,\mathcal{A})$ and let B denotes the incidence matrix of $(V,\mathcal{A})$. Let $\hat{F}_i$ be a completion of F_i with respect to $\mathcal{A}$. Then

$$F_i^c(\alpha) = BX(\alpha), \quad \alpha \in \mathcal{A}_i^c \tag{2.4}$$

by definition (2.2) of the flow complement. Furthermore, if B_i^c denotes a complement of incidence matrix B_i in B then

$$\begin{aligned} B^\tau BX &= [B_i^\tau \ \ (B_i^c)^\tau] \begin{bmatrix} B_i \\ B_i^c \end{bmatrix} X \\ &= B_i^\tau B_i X + (B_i^c)^\tau B_i^c X \\ &= B_i^\tau F_i + (B_i^c)^\tau F_i^c \\ &= [B_i^\tau \ \ (B_i^c)^\tau] \begin{bmatrix} F_i \\ F_i^c \end{bmatrix} \\ &= B^\tau \hat{F}_i \end{aligned}$$

which proves the claim because the left hand side of the equation is the same for both indices.

3 Example

Students were asked to give preference flows, for certain criteria, over the set of their lecturers. The experiment was organized at two different places. At Department of Mathematics (Math) and Department of Psychology (Psycho) of University of Zagreb. Students of Math-group, 29 of them, were allowed to select criteria and alternatives of their own choice, while students of Psycho-group, 48 of them, were forced to select all criteria and all alternatives. Web interface of the questionnaire is placed at URL address `http://pc205.math.hr/Decision/Self/mat` and `http://.../Self/ffzg`. Students were allowed to see only their own ranking after processing their input.

In both cases, the dissimilarity matrix of individual preference flows was calculated for each group and `Statistica 6.0` software was used for clustering.

3.1 Analysis of Math-group

Several clustering methods were consulted and one cluster with two students has been placed on a bit larger distance from the others. The plot is given bellow in Figure 1.

A new Math-group was created without those students and group consensus was calculated again. It happens that one lecturer disappeared from the list, and another one lost two steps in ranking. The others remained their relative position, as shown in Table 1.

After careful examination of the data we discovered that the lecturer who disappeared from the list was used by only one (expelled) student in such a way that he (lecturer) received the highest priority in comparisons with all the others. After inspecting this we considered the student's preference flow tendencious and decided not to consider his data in group consensus.

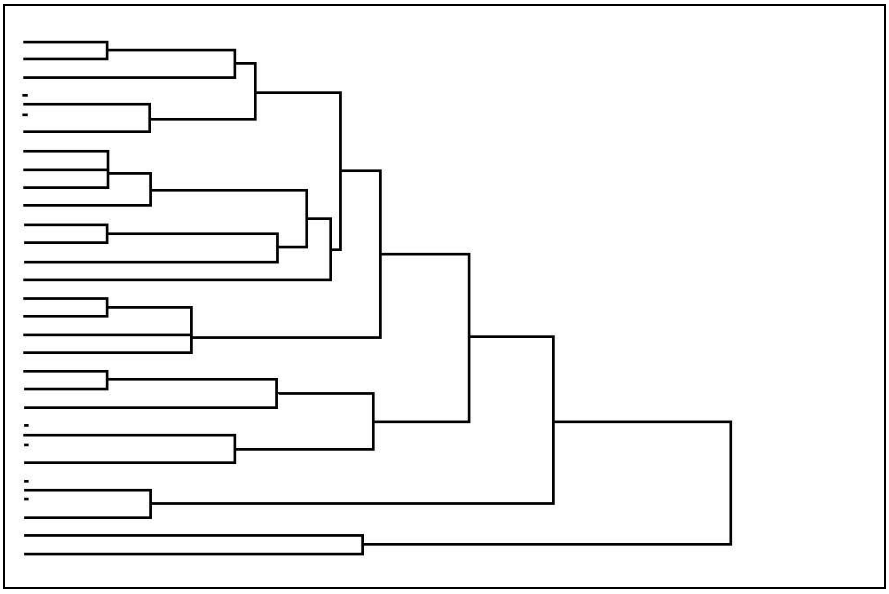

Figure 1: Dissimilarities for Math-group. Weighted pair-group average.

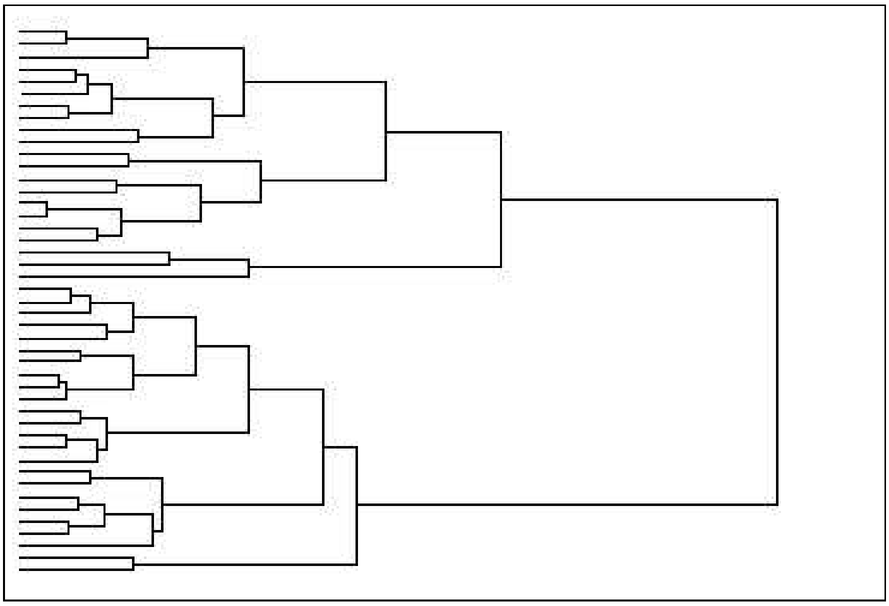

Figure 2: Dissimilarities for Psycho-group. Ward's method.

3.2 Analysis of Psycho-group

As we already said, those students were forced to use all criteria and all alternatives which allow more detailed analysis. In particular, this allows to rank the criteria and make their cluster analysis. In the set of alternatives two clusters of sizes 24 and 21, were present which can be seen from Figure 2.

We calculated criteria ranks for each cluster and for the whole group. Results are given in Table 2.

Table 1: Math-group ranking, full and reduced group.

PROF. NAME	FULL GROUP	REDUCED GROUP	POSITION
John W.	0.109	0.115	the same
Britney S.	0.096		disappeared
Mark A.	0.094	0.100	the same
Pamella A.	0.090	0.100	the same
James B.	0.088	0.097	the same
Max P.	0.082	0.092	the same
Franco N.	0.079	0.086	two steps down
Glace K.	0.079	0.088	one step up
Ursulla A.	0.079	0.087	one step up
James B. jr.	0.073	0.084	the same
Hyder S.	0.072	0.081	the same
Routh M.	0.059	0.070	the same

Table 2: Criteria ranking.

CRITERIA NAME	CLUSTER 1	CLUSTER 2	FULL GROUP
Teaching qualities	0.407	0.363	0.389
Prof. Comp.	0.384	0.322	0.356
Attitude towards students	0.209	0.315	0.255

Ranking of alternatives is given in Table 3. In ranking obtained from cluster 2, V. Flint and P. Beaute changed their positions (boldface) and this is the only difference in rankings generated by those two clusters. The last row shows the group inconsistency for each cluster and full group. A value of inconsistency below 12 deg is acceptable. This means that group consensus flow is 'not far' from the vector space of inconsistent flows or, using the language of the experiment, students in cluster 1 were more consistent than those in cluster 2.

3.2.1 Comparison with Kemeny's median

As shown, criteria ranking obtained for each cluster and for the whole group induces the same linear order on the set of criteria. It is worthwhile to mention that Kemeny's median rule and Condorcet's rule gives the same linear order for full group profile.

Social preference relation given by Kemeny's procedure is given in table 4. In this

Table 3: Psycho-group ranking, by clusters and full group.

PROF. NAME	CLUSTER 1	CLUSTER 2	FULL GROUP	POSITION
V. Mohamad	0.223	0.259	0.243	the same
L. Kekonnen	0.141	0.207	0.171	the same
A. Morgan	0.137	0.158	0.149	the same
G. Jones	0.133	0.149	0.142	the same
A. V. Moore	0.118	0.067	0.092	the same
N. Flint	**0.107**	**0.044**	0.072	
D. Charm	0.079	0.057	0.069	the same
P. Beaute	**0.063**	**0.059**	0.062	
Inconsistency	8.00 deg	11.95 deg	10.17 deg	

Table 4: Kemeny's social preference relation.

$$\begin{tikzcd}
& \text{ProfComp} \arrow[dr, "1"] & \\
\text{AttStud} \arrow[ur, "1"] \arrow[rr, "1"] & & \text{TeachQual}
\end{tikzcd}$$

table, 'AttitStud' stands for Attitude towards students, 'TeachQual' for Teaching qualities and 'ProfComp' for Professional Competence. It is obvious now that social preference relation $\precsim$ is:

$$\text{AttitStud} \precsim \text{ProfComp} \precsim \text{TeachQual}$$

and the same as in Table 2. Low weights of the social preference in Table 3 may lead to conclusion that those qualities are 'almost equaly preffered'. In PM approach they are strongly separated.

4 Conclusions

Several conclusions can be made.

1. The first one concerns the possibilities of doing fine analysis of obtained data. In Psycho-group we were able to perform that thanks to the fact that students were forced to take in account all criteria. In that case it is possible to create *Criteria profile*, i.e. a group with criteria as the group members, and calculate consensus flow for each criterion taking into account all students' preferences. With *Criteria profile*, obtained now, we can measure the distances between criteria and perform

clustering as we have done with *Students profile.* Dissimilarity matrix for *Criteria profile* of Psycho-group is:

	ProfComp	TeachQual	AttitStud
ProfComp	0	4.62	6.06
TeachQual	4.62	0	3.23
AttitStud	6.06	3.23	0

and both clusters have almost the same dissimilarity matrices. This leads to the conclusion that differences between alternatives (lecturers) do not cause different interpretations of criteria in students' minds.

The criteria profile for the Math-group does not exist because some criteria were not selected and, more importantly, each criterion had a different alternative set because of the small size of the group.

2. The second observation concerns inconsistency measure. Group inconsistency is not a valuable information if more than one cluster is present. Generally speaking, it can happen that two clusters have small inconsistency and an inconsistency of full group flow is high, and vice versa. For that purpose the decision maker should perform clustering to see if there is more than one cluster.

3. The third observation concerns the sensitivity of the dissimilarity matrix of the input data. It seems that distance matrices (or better said clustering) are highly sensitive on flow values. This is, at this moment a subjective statement, motivated by data from Table B. Looking at this table, there is no difference between two clusters from the point of view of the decision maker whose interest is to select the best alternative. During this research the other, smaller, clusters were examined and their consensus was calculated. There was no 'great difference' from the full group ranking.

4. If the number of criteria is not high we are suggesting that the questionnaire organizer force the tester to use all criteria in testing. In choosing alternatives for pairwise comparison a tester can have relative freedom, i.e. to choose at least four or more of them.

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