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Space-Time Bilinear Time Series Models and Their Application

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Abstract

Many phenomena are modeled by time series models. Bilinear time series models are useful for series that display shocks such as disease outbreaks or earthquakes. Spatial time series models deal with processes that feature both spatial and temporal dependencies. This paper reviews these models. Then the focus is on spatial-time bilinear (STBL) models for processes exhibiting both shocks and spatial-time dependency. Some properties of STBL models are given and used to identify the parameter orders of the underlying STBL model. Some examples are provided.

1 Introduction

Many phenomena are modeled by time series models broadly defined. These include diseases, environmental data (such as pollution measures), economic trends (time, productivity levels, prices, etc.), meteorological data (temperatures, wind, rainfall, barometric pressures, etc.), geological trends (e.g., earthquakes), oceanographic data (ocean currents, ocean temperatures, etc.), and so on; the list is endless. Let us for the moment focus on any one of these myriad examples, an epidemic consisting of a disease outbreak.

Many (perhaps most) diseases occur in locations that are part of a broader region, rather than occurring in isolation at any particular site. That is, the numbers of occurrences of a disease at a specific site are typically spatially dependent on the numbers at adjacent sites. Clearly, the standard linear time series models which model dependence over time but all at one location, can not deal with this broader context. Also, surveillance for monitoring and control purposes is a mechanism for detecting changes in historical patterns such as would occur in a disease outbreak. Yet, the standard linear models generally are unable to identify such features; rather

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bilinear models are better suited to modeling time series data which feature sudden outbursts.

A bilinear component in the model is especially important when there is a sudden shock to the system. For example, Figure 1 shows such a shock (or outburst, or sudden but temporary change) to a process. [This particular process is that process at one specific site taken from a larger spatially dependent area (not shown)]. More generally, a typical data set would feature both the spatial and disease outbreak aspects and so likely should be modeled by a model containing both spatial and bilinear components.

Therefore, in this work we describe the space-time bilinear (STBL) model of Dai and Billard (1998), and show how this STBL model is an extension of the traditional standard univariate models (fully described in, e.g., Box and Jenkins, 1976) and of the space time linear model of Cliff et al. (1975) and Pfeifer and Deutsch (1980a, b); see Section 5. The use of the STBL model is demonstrated through some illustrative examples in Section 6. First, in Sections 2, 3 and 4, respectively, basic results for standard linear ARMA models, bilinear models, and space time linear models are described briefly.

2 Standard linear ARMA models

The standard linear autoregressive moving average (ARMA) model is given by

$$z(t) = \sum_{i=1}^{p} \phi_i z(t-i) + \sum_{j=1}^{q} \theta_j e(t-j) + e(t) \tag{2.1}$$

where $\{z(t)\}$, $t = 1, 2, \ldots$, is a sequence of observations and $\{e(t)\}$, $t = 1, 2, \ldots$, is a white noise process with $E\{e(t)\} = 0$ and $Var\{e(t)\} = \sigma^2$, and where ϕ_i, $i = 1, \ldots, p$, are the autoregressive parameters and $\theta_j, j = 1, \ldots, q$, are the moving average parameters. Let us assume further that these $\{e(t)\}$ are normally distributed. While this model was first developed in the 1920s (see, e.g., Yule, 1926, and Walker, 1931), it is sometimes referred to as the Box-Jenkins model (a consequence of the publication of their highly successful text which provides considerable detail on the model and it uses, q.v.). We denote this as ARMA (p, q) with p and q representing the autoregressive order and moving average order of the model, respectively.

There are many properties of the models which need to be satisfied in some aspect. Typically, the most important are stationarity and invertibility. For example, take the ARMA $(1, 0)$ model, or simply the AR(1) model, given by

$$z(t) = \phi z(t-1) + e(t). \tag{2.2}$$

For this model to be stationarity, it is required that $|\phi| < 1$. Otherwise, it is nonstationary. When $|\phi| > 1$, we see from (2.2) that the underlying process is explosive. When $|\phi| = 1$, we have

$$z(t) = z(t-1) + e(t),$$

or

$$w(t) \equiv z(t) - z(t-1) = e(t);$$

that is,

$$w(t) = (1-B)z(t), \quad Bz(t) = z(t-1),$$

where B is the backward shift operator.

That is, by differencing, we have transformed the nonstationary $\{z(t)\}$ process into a stationary $\{w(t)\}$ process. In general, we difference d times to produce stationarity. Thus, the general model is denoted as ARIMA (p, d, q). Our discussion herein will henceforth assume the underlying processes are stationary. A pure autoregressive model will be invertible always. In contrast, a pure moving average process is always stationary but requires conditions on its parameters to achieve invertibility; so, e.g., an MA(1) model

$$z(t) = \theta e(t-1) + e(t) \tag{2.3}$$

requires that $|\theta| < 1$ for the model to be invertible. See Box and Jenkins (1976) for a fuller discussion of stationarity, invertibility and other model properties.

Determination of the (p, d, q) which identifies the model is done through the autocorrelation functions defined, at lag k, by

$$\rho_k = \frac{Cov\{z(t), z(t+k)\}}{\sqrt{Var\{z(t)\}Var\{z(t+k)\}}} \tag{2.4}$$

where

$$Cov\{z(t), z(t+k)\} = E\{z(t) - \bar{z}\}\{z(t+k) - \bar{z}\}$$

is the autovariance function at lag k and $\bar{z}$ is the usual average of the observed observations. For each ARMA (p, q) model, we know the theoretical patterns for ρ_k, $k = 1, 2, \ldots$.

For example, for an AR(1) model,

$$\rho_k = \phi^k, \quad k = 1, 2, \ldots, \tag{2.5}$$

that is, ρ_k decays exponentially; while for an MA(q) model,

$$\begin{aligned} \rho_k &\neq 0, \quad k \leq q, \\ &= 0, \quad k > q, \end{aligned} \tag{2.6}$$

that is, ρ_k cuts off at $k = q$.

The partial autocorrelation functions, ϕ_{pp}, have the reverse pattern. By this we mean that while for pure AR(p) models, ρ_k decays exponentially in some manner, the partial autocorrelation functions cuts off at p. Likewise, for a pure MA(q) model, the partial autocorrelation function decays. For the mixed ARMA (p, q) model, both the autocorrelation functions and the partial autocorrelation functions decay (and do not cut off).

These theoretical autocorrelation and partial autocorrelation functions are therefore compared with the sample autocorrelations $\hat{\rho}_k$, $k = 1, 2, \ldots$, and the sample partial autocorrelation functions, calculated from the data, to identify tentative values for p, q and d. If these plots of $\hat{\rho}_k$ as k increases do not decay towards zero, then the data need further differencing to produce stationarity.

3 Bilinear models

The bilinear autoregressive moving average (BL) model added a cross-product term to the standard linear ARMA model to give

$$\begin{aligned} z(t) &= \sum_{i=1}^{p} \phi_i z(t-i) + \sum_{j=1}^{q} \theta_j e(t-j) \\ &\quad + \sum_{i=1}^{r} \sum_{j=1}^{s} \beta_{ij} z(t-i) e(t-j) + e(t), \end{aligned} \tag{3.1}$$

and is denoted by BL(p, q, r, s). This model is linear in $z(t)$ alone and in $e(t)$ alone, but not in both. It was first developed by Mohler (1973) for control theory and is particularly applicable for time series data exhibiting shocks. For example, the BL(1,1,1,1) model is

$$z(t) = \phi z(t-1) + \theta e(t-1) + \beta z(t-1)e(t-1) + e(t); \tag{3.2}$$

and the pure bilinear model BL(0,0,1,1) is

$$z(t) = \beta z(t-1)e(t-1) + e(t). \tag{3.3}$$

Quinn (1982), Pham and Tran (1981) and Bhaskara Rao et al. (1983) look at stationarity conditions for specific models, while Liu and Brockwell (1988) and Liu (1989, 1992) look at a general bilinear model. Invertibility issues were considered by Granger and Andersen (1978), Subba Rao (1981), Liu (1985) and Quinn (1982) for particular models.

Identification of BL models using the autocorrelations functions analogously to the approach for ARMA models does not work satisfactorily since, as Granger and Andersen (1978) observed, the second order autocorrelations for the model

$$z(t) = \beta z(t-k)e(t-l) \tag{3.4}$$

are zero, leading incorrectly to an identification of a white noise process or a linear model. Granger and Andersen (1978) therefore suggested that the autocorrelation functions of the squared observations $\{z^2(t)\}$ be used instead. Li (1984) thus calculated the theoretical values for some simple models.

4 Space time ARMA models

The standard linear ARMA model represents a series over time at a single location. There are many situations, e.g., the spread of disease, pollution, social measures such as income, for which there is a dependence across sites within a spatially contiguous location as well as temporal dependence. In these cases, rather than fitting separate linear ARMA models to each site which cannot capture any spatial dependence, it is desirable to fit a model which accommodates both spatial and temporal dependence. Working within a social geography framework, Cliff et al. (1975) and Pheifer and Deutsch (1980a, b) proposed an extension of the temporal linear ARMA model of (2.1) to the space time autoregressive moving average (STARMA) model given by

$$\boldsymbol{z}(t) = \sum_{i=1}^{p} \sum_{m=0}^{\lambda_i} \phi_m^i \boldsymbol{W}^{(m)} \boldsymbol{z}(t-i) + \sum_{j=1}^{q} \sum_{n=0}^{\eta_j} \theta_n^j \boldsymbol{W}^{(n)} \boldsymbol{e}(t-j) + \boldsymbol{e}(t) \tag{4.1}$$

where $\boldsymbol{z}(t) = [z_1(t), \ldots, z_g(t)]^T$ represents the observation at time t at each of g spatial sites, $\boldsymbol{e}(t) = [e_1(t), \ldots, e_g(t)]^T$ is the white noise process for the g sites, and where λ_i is the spatial order of the autoregressive term at temporal lag i, η_j is the spatial order of the moving average term at temporal lag j, and $\boldsymbol{W}^{(m)} = (w_{ku}^{(m)})$ is the $g \times g$ weighting matrix for spatial order m. We write $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_p)$ and $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_q)$ and denote the model by STARMA$(p_{\boldsymbol{\lambda}}, q_{\boldsymbol{\eta}})$.

The elements $w_{ku}^{(m)}$ of the weight matrix $\boldsymbol{W}^{(m)}$ represent the mth order spatial weight for neighbor site u of the kth site. As an illustration, consider the spatial region with nine sites on a regular grid as indicated in the figure below:

1	2	3
4	5	6
7	8	9

Let us suppose that equal weights are assigned to each neighbor. Then, the first $(m = 1)$ order weight matrix corresponds to that given in Table 1a. For example, site 2 has as first order sites 1, 3 and 5. Therefore, $w_{21}^{(1)} = w_{23}^{(1)} = w_{25}^{(1)} = 1/3$ and all other weights in that row $(k = 2$ row) are zero. Then, site 2 is first order spatially dependent on the other sites according to

$$\{\boldsymbol{W}^{(1)}\boldsymbol{z}(t)\}_{k=2} = (1/3)z_1(t) + 0z_2(t) + (1/3)z_3(t) + 0z_3(5) + 0z_4(t) + (1/3)z_5(t) + \ldots + 0z_9(t),$$

i.e.

$$z_2(t) = (1/3)z_1(t-1) + (1/3)z_3(t-1) + (1/3)z_5(t-1) \tag{4.2}$$

and is clearly a weighted sum of its neighbors. The second order weight matrix $\boldsymbol{W}^{(2)}$ is given in Table 1b. The weights can be arbitrary, with the only condition being that the row sums satisfy

$$\sum_{u=1}^{g} w_{ku}^{(m)} = 1 \tag{4.3}$$

Table 1a: First-order weight matrix on 3×3 grid: $\boldsymbol{W}^{(1)}$.

	Neighbor								
Site	1	2	3	4	5	6	7	8	9
1	.*	1/2	.	1/2	.	.	.	.	.
2	1/3	.	1/3	.	1/3	.	.	.	.
3	.	1/2	.	.	.	1/2	.	.	.
4	1/3	.	.	.	1/3	.	1/3	.	.
5	.	1/4	.	1/4	.	1/4	.	1/4	.
6	.	.	1/3	.	1/3	.	.	.	1/3
7	.	.	.	1/2	.	.	.	1/4	.
8	.	.	.	.	1/3	.	1/3	.	1/3
9	.	.	.	.	.	1/2	.	1/2	.

* An '.' represents $w_{ku}^{m} = 0$.

Table 1b: Second-order weight matrix on 3×3 grid: $\boldsymbol{W}^{(2)}$.

	Neighbor								
Site	1	2	3	4	5	6	7	8	9
1	.	.	.	.	1	.	.	.	.
2	.	.	.	1/2	.	1/2	.	.	.
3	.	.	.	.	1	.	.	.	.
4	.	1/2	.	.	.	.	.	1/2	.
5	1/4	.	1/4	.	.	.	1/4	.	1/4
6	.	1/2	.	.	.	.	.	1/2	.
7	.	.	.	.	1	.	.	.	.
8	.	.	.	1/2	.	1/2	.	.	.
9	.	.	.	.	1	.	.	.	.

for each k and each m. When $m = 0$, we set $\boldsymbol{W}^{(0)} = \boldsymbol{I}$, the identity matrix; i.e. there is no spatial dependence. The weights are assumed to be known. It is not necessary that the sites be on a regular grid. Typically, sites will be irregularly spaced as when each site represents (say) a country or state within the larger contiguous region under study.

We observe that when $g = 1$ and $W^{(m)} = 1$ for all m, the model (4.1) simplifies to the standard ARMA model of a single site. It is also possible to show that if $\boldsymbol{W}^{(m)}$ is such that it has one element equal to 1 (appropriately identified) and the remaining $(g \times g - 1)$ elements equal to zero, the model (4.1) becomes the multiple

ARMA model where instead of $\boldsymbol{z}(t)$ being the vector of observations at g sites it now represents a multivariate variate with g components (but at one site). This multiple ARMA model is a special case of the multitple bilinear model of Stensholt and Tjoshteim (1987).

Pfeifer and Deutsch (1980b) investigated identification issues for the STARMA model. They derived the autocorrelation function for a STARMA $(0,1_1)$ model and gave a detailed analysis of the patterns that prevail for different scenarios for that model and also for the STARMA$(1_1,\ 0)$ model. Further, they provided general pattern characteristics of the theoretical autocorrelation and partial autocorrelation functions.

Thus, the space-time autocorrelation function for h spatial lags and j time lags apart is given by

$$\rho_h(j) = \tau_{h0}(j)/\sqrt{\tau_{hh}(0)\tau_{00}(0)} \tag{4.4}$$

where $\tau_{hk}(j)$ is the space-time autocorvariance function between the hth and kth order neighbors and j time lags apart is

$$\begin{aligned}\tau_{hk}(j) &= (1/g)Cov\{\boldsymbol{W}^{(h)}\boldsymbol{z}(t), \boldsymbol{W}^{(k)}\boldsymbol{z}(t-j)\} \\ &= (1/g)tr\{[\boldsymbol{W}^{(k)}]^T\boldsymbol{W}^{(h)}\boldsymbol{\Gamma}(j)\}\end{aligned} \tag{4.5}$$

with

$$\boldsymbol{\Gamma}(j) = Cov\{\boldsymbol{z}(t),\quad \boldsymbol{z}(t-j)\} \tag{4.6}$$

and $tr\boldsymbol{A}$ is the trace of the matrix $\boldsymbol{A}$. The space-time partial autocorrelation function is the coefficient ϕ'_{kl} obtained from solving the system of equations

$$\tau_{h0}(j) = \sum_{i=1}^{k}\sum_{l=1}^{\lambda}\phi'_{jl}\tau_{hl}(j-i) \tag{4.7}$$

as $l = 0, 1, \ldots, \lambda$ for $k = 1, 2, \ldots$, in turn.

For the STARMA $(p_{\boldsymbol{\lambda}}, 0)$ model, the autocorrelation function decays while the partial autocorrelation function cuts off after p time lags and λ_p spatial lags. For the STARMA $(0,\ q_{\boldsymbol{\eta}})$ model, the autocorrelation function cuts off after q time lags and η_q spatial lags while the partial autocorrelation function decays. For the mixed STARMA $(p_{\boldsymbol{\lambda}}, q_{\boldsymbol{\eta}})$ model, both functions decay to zero. Notice the similarities between these patterns for the space time linear models with their counterparts for the non-spatial temporal-only linear ARMA models.

5 Space time bilinear models

When there are both spatial and time dependence and shocks present in the system, we want to combine the features of the STARMA and the BL models.

Therefore, from Dai and Billard (1998), we have the space time bilinear (STBL) model, given for $k = 1, \ldots, g$, $t = 1, 2, \ldots$, by

$$\begin{aligned} z_k(t) &= \sum_{i=1}^{p}\sum_{m=0}^{\lambda_i} \phi_m^j \{\sum_{u=1}^{g} w_{ku}^{(m)} z_u(t-i)\} + e_k(t) \\ &+ \sum_{j=1}^{g}\sum_{n=0}^{\eta_j} \theta_n^j \{\sum_{v=1}^{g} w_{kv}^{(n)} e_v(t-h)\} \\ &+ \sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{m=0}^{\xi_i}\sum_{n=0}^{\mu_j} \beta_{mn}^{ij} \{\sum_{u=1}^{g} w_{ku}^{(m)} z_u(t-1)\}\{\sum_{v=1}^{g} w_{kv}^{(n)} e_v(t-j)\}, \end{aligned} \tag{5.1}$$

or, in its vector form, by

$$\begin{aligned} \boldsymbol{z}(t) = \sum_{i=1}^{p}\sum_{m=0}^{\lambda_i} \phi_m^i \boldsymbol{W}^{(m)} \boldsymbol{z}(t-i) + \sum_{j=1}^{q}\sum_{n=0}^{\eta_j} \theta_n^j \boldsymbol{W}^{(n)} \boldsymbol{e}(t-j) \\ + \sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{m=0}^{\xi_i}\sum_{n=0}^{\mu_j} \beta_{mn}^{ij} [\boldsymbol{W}^{(m)} \boldsymbol{z}(t-i)] \# \boldsymbol{e}(t-j)] + \boldsymbol{e}(t), \end{aligned} \tag{5.2}$$

where p is the autoregressive order, q is the moving average order, r is the autoregressive order in the bilinear term, s is the moving average order in the bilinear term, λ_i is the spatial order of the autoregressive term at temporal lag i, η_j is the spatial order of the moving average term at temporal lag j, ξ_i is the spatial order of the autoregressive term in the bilinear term at temporal lag i, μ_j is the spatial order of the moving average term in the bilinear term at temporal lag j, ϕ_m^i is the autoregressive parameter at temporal lag i and spatial lag m, θ_n^j is the moving average parameter at temporal lag j and spatial lag n, β_{mn}^{ij} is the bilinear parameter at temporal lags i and j for the autoregressive and the moving average terms, respectively, and at spatial lags m and n for the autoregressive and moving average terms, respectively, $\boldsymbol{W}^{(m)} = (w_{ku}^{(m)})$ is the $g \times g$ weighting matrix at spatial order m, and $\boldsymbol{e}(t) = [e_1(t), \ldots, e_g(t)]^T$ is a sequence of independent identically distributed vector random variables with

$$\begin{aligned} E[\boldsymbol{e}(t)] &= \boldsymbol{0}, \\ E[\boldsymbol{e}(t)\boldsymbol{e}(t+j)^T] &= \begin{cases} \boldsymbol{G}, & j = 0, \\ \boldsymbol{0}, & j \neq 0, \end{cases} \\ E[\boldsymbol{z}(t)\boldsymbol{e}(t+j)^T] &= \boldsymbol{0}, \qquad j > 0. \end{aligned}$$

We write $A\#B = (c_{ij})$, where $c_{ij} = a_{ij}b_{ij}$, is defined as the matrix element-wise multiplication for any matrices $A = (a_{ij})$ and $B = (b_{ij})$ of the same size. Analogously to the bilinear and STARMA models, we denote (5.2) as the STBL$(p_{\boldsymbol{\lambda}}, q_{\boldsymbol{\eta}}, r_{\boldsymbol{\xi}}, s_{\boldsymbol{\mu}})$ model of temporal order p, q, r, and s; and spatial order $\boldsymbol{\lambda} = [\lambda_1, \ldots, \lambda_p]$, $\boldsymbol{\eta} = [\eta_1, \ldots, \eta_q]$, $\boldsymbol{\xi} = [\xi_1, \ldots, \xi_r]$, and $\boldsymbol{\mu} = [\mu_1, \ldots, \mu_s]$.

For example, the STBL $(0,0,1_1,1_0)$ model where the weights are those of Table 1 is, for site $k=1$,

$$z_1(t) = \beta_{00}z_1(t-1)e_1(t-1)+\beta_{10}[(1/2)z_2(t-1)+(1/2)z_4(t-1)]e_1(t-1)+e_1(t) \quad (5.3)$$

where for notational simplicity β^{11}_{mn} is written as β_{mn}. Notice that the moving average part of the bilinear term has no spatial dependence, since $\mu=0$. If, however, we add spatial dependence to this part of the model to give the STBL$(0,0,1_1,1_1)$ model, then, at site $k=1$, we have

$$\begin{aligned} z_1(t) &= \beta_{00}z_1(t-1)e_1(t-1)+\beta_{10}[\frac{1}{2}z_2(t-1)+\frac{1}{2}z_4(t-1)]e_1(t-1) \\ &+ \beta_{01}z_1(t-1)[\frac{1}{2}e_2(t-1)+\frac{1}{2}e_4(t-1)] \\ &+ \beta_{11}[\frac{1}{2}z_2(t-1)+\frac{1}{2}z_4(t-1)][\frac{1}{2}e_2(t-1)+\frac{1}{2}e_4(t-1)]+e_1(t). \end{aligned} \quad (5.4)$$

For the general model, there are three special cases of importance.

(i) If $\beta^{ij}_{mn}=0$, for all i,j,m, and n, then the STARMA model of (4.1) is recovered.

(ii) If $g=1$ and $W^{(m)}=1$ for all m, then the BL model of (3.1) is recovered.

(iii) If $\beta^{ij}_{mn}=0$ for all i,j,m, and n, and if $g=1$ and $W^{(m)}=1$ for all m, then we capture the classical ARMA model of (2.1).

Dai and Billard (1998) also show that the STBL model can be structured as a multiple bilinear model. However, the STBL model has

$$P_1=\sum_{i=1}^{p}(\lambda_i+1)+\sum_{j=1}^{q}(\eta_j+1)+\{\sum_{i=1}^{r}(\xi_i+1)\}\{\sum_{j=1}^{s}(\mu_j+1)\} \quad (5.5)$$

parameters, whereas the multiple bilinear model has $P_2=g^2(p+q+grs)$ parameters; see Stensholt and Tjostheim (1987). It can be shown that $P_1<P_2$. To illustrate, when $p=q=r=s=1$ and $\lambda=\eta=\xi=\mu=1$, a not unreasonable common case, we have

$$P_1=8 \quad \text{and} \quad P_2=g^2(2+g).$$

Then, $P_1>P_2$ only if $g^2(2+g)>8$, which implies $P_1>P_2$ only if $g=1$. However, when $g=1$, we have one site and the model is no longer spatial. This reduced number of parameters for the STBL model makes it an attractive alternative to the multiple bilinear model.

In addition to the aforementioned special cases which recapture only more restrictive classes of models, there are four other special cases within the class of STBL models itself which will be particularly useful as the model's behavior is studied. These are:

(a) The diagonal model is when $\beta_{mn}^{ij} = 0$, $i \neq j$.

(b) The subdiagonal model is when $\beta_{mn}^{ij} = 0$, $i < j$.

(c) The superdiagonal model is when $\beta_{mn}^{ij} = 0$, $i > j$.

(d) The pure bilinear model is when

$$\boldsymbol{a}^i = \sum_{m=0}^{\lambda_i} \phi_m^i \boldsymbol{W}^{(m)} = 0, \quad \boldsymbol{b}^j = \sum_{n=0}^{\eta_j} \theta_n^j \boldsymbol{W}^{(n)} = 0, \quad \text{for all} \quad i, j.$$

The identification procedure for the STBL model has two stages. First, the sample autocorrelation and sample partial autocorrelation functions are calculated and compared with the patterns for the corresponding theoretical functions as though a linear STARMA model were being fitted. This allows identification, at least tentatively, of the $p_{\boldsymbol{\lambda}}$ and $q_{\boldsymbol{\eta}}$ orders. The tentative STARMA $(p_{\boldsymbol{\lambda}}, q_{\boldsymbol{\eta}})$ model is then fitted to the data and the residuals calculated. If the $p_{\boldsymbol{\lambda}}$ and $q_{\boldsymbol{\eta}}$ values are correct or nearly so, then the residuals should be a pure bilinear model (case (d) above).

As observed by Granger and Andersen (1978) for the temporal bilinear model, so here is it necessary to work with the squared observations when determining the autocorrelation functions. Following their lead and by analogy with those of Pfeifer and Deutsch (1980b), Dai and Billard (1998) give definitions for the space-time autocorrelation function at spatial lag h and time lag j equivalent to those of (4.4)-(4.6) but with $\boldsymbol{z}(t)$ everywhere replaced by $\boldsymbol{z}^2(t)$. Likewise, the partial autocorrelation function of (4.7) pertains but applied to the squared observations $\boldsymbol{z}^2(t)$. They derived expressions for the space-time covariance functions for several special multiple bilinear models, from which those for some STBL models follow.

Therefore, for the diagonal STBL model

$$\boldsymbol{z}(t) = \sum_{i=1}^{p} \sum_{m=0}^{\xi} \sum_{n=0}^{\mu} \beta_{mn} \{\boldsymbol{W}^{(m)} \boldsymbol{z}(t-i)\} \# \{\boldsymbol{W}^{(n)} \boldsymbol{e}(t-i)\} + \boldsymbol{e}(t),$$

the space-time autocorrelation function of $\boldsymbol{z}^2(t)$ satisfies

$$\begin{aligned} \rho_h(j) &= 0, \quad \text{if} \quad h > \max(\xi, \mu), &(5.6)\\ \rho_h(j) &= 0, \quad \text{if} \quad \beta_{hh} = 0 \quad \text{for} \quad h \leq \max(\xi, \mu). &(5.7) \end{aligned}$$

For the subdiagonal STBL model

$$\boldsymbol{z}(t) = \sum_{i=1}^{p} \sum_{\substack{j=1 \\ j>i}}^{q} \sum_{m=0}^{\xi} \sum_{n=0}^{\mu} \beta_{mn} \{\boldsymbol{W}^{(m)} \boldsymbol{z}(t-i)\} \# \{\boldsymbol{W}^{(n)} \boldsymbol{e}(t-j)\} + \boldsymbol{e}(t),$$

the space-time autocorrelation function of $\boldsymbol{z}^2(t)$ satisfies

$$\rho_h(j) = 0, \quad \text{if} \quad h > \max(\boldsymbol{\xi}, \boldsymbol{\mu}), \tag{5.8}$$

$$\rho_h(j) = 0, \quad \text{if} \ \ \beta_{mn} = 0 \ \ \text{if } m \text{ or } n \text{ equals } h \leq \max(\boldsymbol{\xi}, \boldsymbol{\mu}). \tag{5.9}$$

These results show there is a natural cutoff for spatial lag h at $\max(\boldsymbol{\xi}, \boldsymbol{\mu})$. They also imply that there is no general cutoff point in spatial lag h for fixed time lag j (though it can occur in some cases). Another implication is that in $\rho_h(j)$ the time lag j can be fixed and that we need only observe the pattern in the spatial lag h.

6 Illustrative examples

The foregoing STBL model results are illustrated through the following three examples. In each case, data were generated on a 5×5 spatial grid with mth order, $m = 1, 2, 3$, spatial weights being those obtained by assuming all mth order neighbors have the same weights $w_{ku}^{(m)}$ for each m and u; that is, the weights are the analogue of those given in Table 1 for a 3×3 grid.

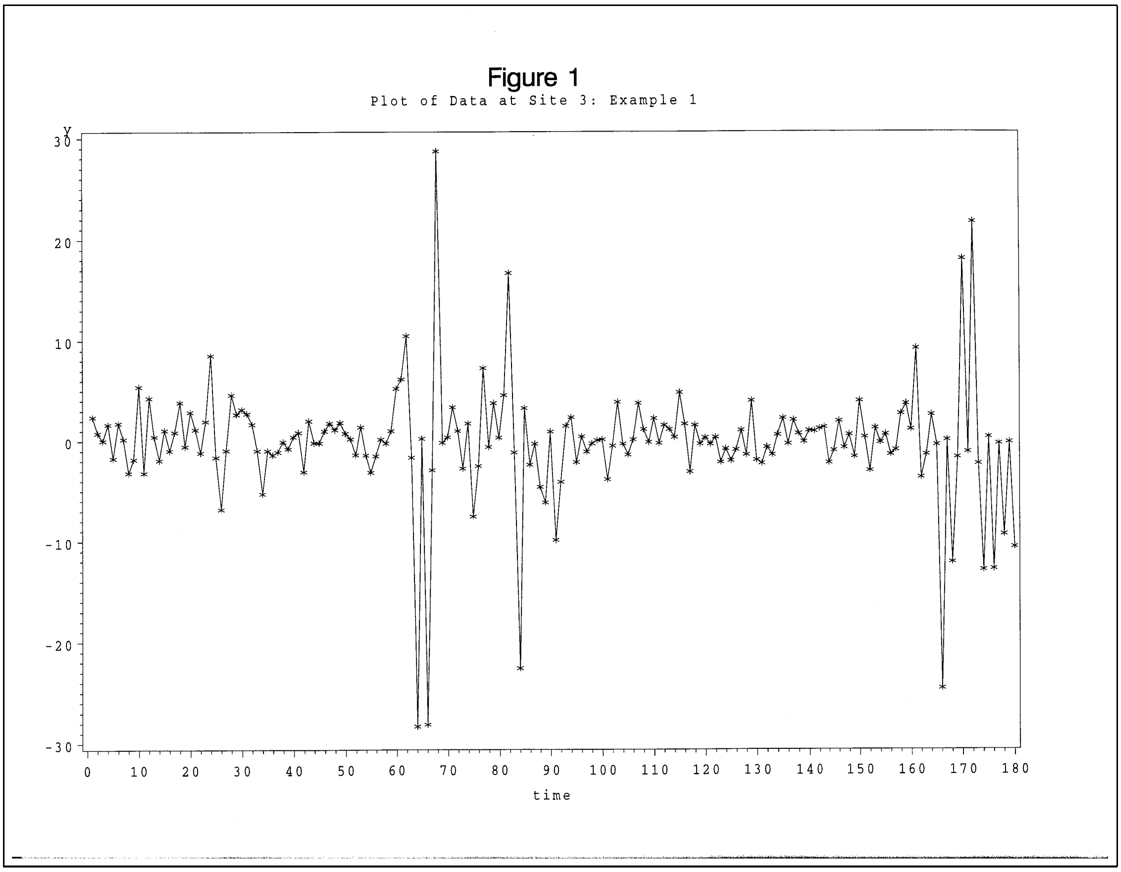

Figure 1: Plot of data at site 3, Example 1.

Example 1.

The first example consists of a set of data generated from a STBL $(0, 0, 2_{\boldsymbol{\xi}}, 4_{\boldsymbol{\mu}})$ model with $\boldsymbol{\xi} = (0, 1)$ and $\boldsymbol{\mu} = (0, 0, 0, 2)$. This is a pure bilinear model involving

Table 2: Sample space-time autocorrelation functions for $\{\boldsymbol{z}(t)\}$: Example 1.

		Spatial lag			
	$\rho_h(j)$	$h=0$	$h=1$	$h=2$	$h=3$
	$j=1$	-.005	0.060	0.011	-.008
Time	$j=2$	-.006	0.035	-.086	0.020
lag	$j=3$	0.011	0.007	-.018	0.006
	$j=4$	-.176	-.125	-.035	-.012
	$j=5$	-.003	0.009	0.019	-.003
	$j=6$	-.039	0.034	-.009	0.029
	$j=7$	-.022	-.006	0.007	-.010
	$j=8$	0.076	0.020	0.015	-.015

only the cross-product terms and β_{mn}^{ij} parameters, in equation (5.1). The parameter values used were

$$\beta = \begin{pmatrix} \beta_{00}^{11} & \beta_{00}^{12} & \beta_{00}^{13} & \beta_{00}^{14} & \beta_{01}^{14} & \beta_{02}^{14} \\ \beta_{00}^{21} & \beta_{00}^{22} & \beta_{00}^{23} & \beta_{00}^{24} & \beta_{01}^{24} & \beta_{02}^{24} \\ \beta_{10}^{21} & \beta_{10}^{22} & \beta_{10}^{23} & \beta_{10}^{24} & \beta_{11}^{24} & \beta_{12}^{24} \end{pmatrix} = \begin{pmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.5 & 0.0 & -.4 & 0.0 & 0.3 \\ 0.0 & 0.7 & 0.5 & -.7 & 0.0 & 0.9 \end{pmatrix} \quad (6.1)$$

The plot of Figure 1 represents these data points over time at site three.

Table 2 displays the sample space-time autocorrelation functions obtained by using the sample counterparts of equations (4.4)-(4.6) on the observations $\{\boldsymbol{z}(t)\}$. The absence of any pattern among these values suggest there are no linear terms in the model, confirming the $p = q = 0$ values.

Therefore, the second stage involving the squared observations is invoked. Thus, the sample space-time autocorrelation functions for $\{\boldsymbol{z}^2(t)\}$ are calculated, and shown in Table 3. At time lags $j = 2$ and $j = 4$, the sample space-time autocorrelations are non-zero.

These reflect the (known) $r = 2$ and $s = 4$ values, respectively. The values for $\rho_h(j)$ cut off at spatial lag $h = 2$, as expected since $\rho_h(j) = 0$ for $h > \max(\xi, \mu) = 2$. It is also the case that both the sample space-time autocorrelation and partial autocorrelation functions (not shown) tend to decay exponentially. This reflects the presence of both autoregressive and moving average terms in the model.

Example 2

The second data set was generated from the model STBL $(1_1, 0, 3_{\boldsymbol{\xi}} 1_1)$ with $\boldsymbol{\xi} =$

Table 3: Sample space-time autocorrelation functions for $\{\boldsymbol{z}^2(t)\}$: Example 1.

	$\rho_h(j)$	Spatial lag $h=1$	$h=2$	$h=3$	$h=4$
	$j=1$	-.022	-.015	-.002	-.010
Time	$j=2$	**0.368**	**0.490**	**0.122**	0.025
lag	$j=3$	-.022	-.015	-.005	-.009
	$j=4$	**0.230**	**0.214**	**0.132**	0.099
	$j=5$	-.022	-.016	-.005	-.009
	$j=6$	0.163	0.120	0.057	0.079
	$j=7$	-.021	-.017	-.005	-.008
	$j=8$	0.132	0.129	0.022	0.037

$(0,0,2)$ and parameter values $\boldsymbol{\phi} = (\phi_0^1, \phi_1^1) = (0.5, -0.4)$, and

$$\boldsymbol{\beta} = \begin{pmatrix} \beta_{00}^{11} & \beta_{01}^{11} \\ \beta_{00}^{21} & \beta_{01}^{21} \\ \beta_{00}^{31} & \beta_{01}^{31} \\ \beta_{10}^{31} & \beta_{11}^{31} \\ \beta_{20}^{31} & \beta_{21}^{31} \end{pmatrix} = \begin{pmatrix} 0.0 & 0.4 \\ 0.0 & -.3 \\ 0.0 & 0.7 \\ 0.0 & 0.0 \\ 0.0 & 0.8 \end{pmatrix}. \tag{6.2}$$

Tables 4a and 4b display the sample autocorrelation and partial autocorrelation functions for the data itself $\{\boldsymbol{z}(t)\}$. From Table 4a, the exponential decay across time and spatial lags is evident. Since from Table 4b we see that there is a cutoff in the sample space-time partial autocorrelation function at time lag $j=1$ and spatial lag $h=1$, we conclude that the (linear) STARMA model would be tentatively identified as STARMA $(1_1, 0)$. That is, the linear autoregressive term of the model includes a first order spatial $(\lambda = 1)$ component. [If there were a standard linear (but nonspatial) autoregressive term only (applicable if $\lambda = 0$) as in a standard AR(1) model of equation (2.2), then the spatial cutoff in Table 4b would have been at $h=0$ instead of at $h=1$.]

In order to identify any bilinear terms in the model, we first fit a STARMA $(1_1, 0)$ model to the data, and then study the resulting residuals $\{\boldsymbol{x}(t)\}$. These residuals should represent data from the STBL $(0, 0, 3_{\boldsymbol{\xi}} 1_1)$ model with $\boldsymbol{\xi}$ and $\boldsymbol{\beta}$, as above in equation (6.2). To identify the model orders of these residuals, we need to calculate the sample space-time autocorrelation and partial autocorrelation functions of the squared residuals $\{\boldsymbol{x}^2(t)\}$. The autocorrelation functions are displayed in Table 5. It is observed that the spatial cutoff is at $h=2$, consistent with $\rho_h(j) = 0$ for $h > \max(\xi, \mu) = 2$. Likewise, a time lag of $r=3$ is identified by the nonzero values at $j=3$ for both $h=0$ and $h=2$; that is, $\rho_0(3) = 0.339$ and $\rho_2(3) = 0.169$.

The "zero" value for $\rho_1(3) = .001 \approx 0$ pertains since for $h=1$, the $\beta_{11}^{31} = 0$,

Table 4a: Sample space-time autocorrelation functions of $\{\boldsymbol{z}(t)\}$: Example 2.

	$\rho_h(j)$	Spatial lag $h=0$	$h=1$	$h=2$	$h=3$
	$j=1$	0.569	-.474	0.447	0.231
Time	$j=2$	0.423	-.439	0.353	0.223
lag	$j=3$	0.420	-.433	0.343	0.221
	$j=4$	0.281	-.357	0.278	0.172
	$j=5$	0.220	-.317	0.220	0.171
	$j=6$	0.250	-.240	0.250	0.195
	$j=7$	0.169	-.189	0.185	0.144
	$j=8$	0.134	-.172	0.155	0.145

Table 4b: Sample space-time partial autocorrelation functions of $\{\boldsymbol{z}(t)\}$: Example 2.

	ϕ'_{hj}	Spatial lag $h=0$	$h=1$	$h=2$	$h=3$
	$j=1$	0.569	-.362	0.075	-.083
Time	$j=2$	0.055	-.120	-.066	0.092
lag	$j=3$	0.139	-.158	-.026	-.023
	$j=4$	-.154	0.063	0.001	-.057
	$j=5$	-.025	-.095	-.027	0.033
	$j=6$	0.047	0.098	0.069	0.015
	$j=7$	-.042	0.034	-.010	-.019
	$j=8$	0.006	-.025	0.048	0.025

as expected from equation (5.9). Suppose instead the data set had been generated with the parameters

$$\beta = \begin{pmatrix} 0.0 & 0.4 \\ 0.0 & -.3 \\ 0.0 & 0.7 \\ 0.0 & 0.6 \\ 0.0 & 0.8 \end{pmatrix}; \tag{6.3}$$

that is, the same parameter values as in equation (6.2) except that now $\beta_{11}^{31} = 0.6$. Then, after fitting the linear STARMA $(1_1, 0)$ model as before; we obtain the sample space-time autocorrelation function of the resulting squared residuals $\{\boldsymbol{x}^2(t)\}$ to be as shown in Table 6. As expected, the same patterns emerge indicating a spatial lag of 2 and a time lag of 3, except that now the $\rho_1(3) = 0.114$ value is no longer zero.

Table 5: Sample space-time autocorrelation functions of $\{\boldsymbol{x}^2(t)\}$: Example 2.

	$\rho_h(j)$	Spatial lag $h=0$	$h=1$	$h=2$	$h=3$
	$j=1$	0.094	0.110	0.036	0.018
Time	$j=2$	0.100	0.026	0.012	-.005
lag	$j=3$	**0.338**	0.001	**0.169**	0.056
	$j=4$	0.097	0.059	0.031	0.002
	$j=5$	0.058	0.033	0.023	0.005
	$j=6$	0.166	0.012	0.073	0.011
	$j=7$	0.044	0.016	0.038	0.017
	$j=8$	0.070	0.027	0.012	0.003

Table 6: Sample space-time autocorrelation functions for $\{\boldsymbol{x}^2(t)\}$: Example 2$'$.

	$\rho_h(j)$	Spatial lag $h=0$	$h=1$	$h=2$	$h=3$
	$j=1$	0.136	0.132	0.053	0.033
Time	$j=2$	0.144	0.047	0.023	0.007
lag	$j=3$	**0.379**	**0.114**	**0.168**	0.020
	$j=4$	0.184	0.066	0.081	0.019
	$j=5$	0.100	0.030	0.046	0.022
	$j=6$	0.247	0.078	0.075	0.008
	$j=7$	0.065	0.039	0.035	0.022
	$j=8$	0.101	0.034	0.026	0.016

Example 3

Let us now consider a data set generated from a STBL (0, 0, $2_{\boldsymbol{\xi}}, 2_{\boldsymbol{\mu}}$) model with $\boldsymbol{\xi} = (0,2)$ and $\boldsymbol{\mu}(1,1)$, and parameters

$$\boldsymbol{\beta} = \begin{pmatrix} \beta_{00}^{11} & \beta_{01}^{11} & \beta_{00}^{12} & \beta_{01}^{12} \\ \beta_{00}^{21} & \beta_{01}^{21} & \beta_{00}^{22} & \beta_{01}^{22} \\ \beta_{10}^{21} & \beta_{11}^{21} & \beta_{10}^{22} & \beta_{11}^{22} \\ \beta_{20}^{21} & \beta_{21}^{21} & \beta_{20}^{22} & \beta_{21}^{22} \end{pmatrix} = \begin{pmatrix} 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.8 & -.4 \\ 0.0 & 0.0 & 0.3 & 0.7 \\ 0.0 & 0.0 & 0.6 & 0.9 \end{pmatrix}. \qquad (6.4)$$

Since $\beta_{mn}^{ij} = 0$ for all $i \neq j$, the underlying model is a diagonal STBL model.

The sample space-time autocorrelation functions for the $\{\boldsymbol{z}^2(t)\}$ are given in Table 7a. There is a clear spatial cutoff at lag $h = 2$ which matches the $\max(\xi, \mu) = 2$ value. This is particularly noticable at time lags $j = 2$, 4, 6. These autocorrelation values decay over time lags (at each $h = 0$, 1, 2). When we look at the sample

Table 7a: Sample space-time autocorrelation functions for $\{\boldsymbol{z}^2(t)\}$: Example 3.

	$\rho_h(j)$	Spatial lag $h=0$	$h=1$	$h=2$	$h=3$
	$j=1$	-.019	-.018	-.018	-.015
Time	$j=2$	**0.468**	**0.321**	**0.489**	0.158
lag	$j=3$	-.019	-.018	-.018	-.015
	$j=4$	**0.123**	**0.232**	**0.421**	0.155
	$j=5$	-.019	-.018	-.018	-.015
	$j=6$	**0.077**	**0.186**	**0.185**	0.209
	$j=7$	-.019	-.018	-.018	-.015
	$j=8$	0.039	0.154	0.104	0.169

Table 7b: Sample space-time partial autocorrelation functions for $\{\boldsymbol{z}^2(t)\}$: Example 3.

	ϕ'_{hj}	Spatial lag $h=0$	$h=1$	$h=2$	$h=3$
	$j=1$	-.0187	-.020	-.109	-.015
Time	$j=2$	**0.468**	**0.277**	**0.710**	0.021
lag	$j=3$	-.003	-.001	0.004	-.000
	$j=4$	-.196	-.130	0.099	-.006
	$j=5$	-.005	-.002	0.005	-.000
	$j=6$	0.031	0.150	-.021	0.110
	$j=7$	-.004	0.000	0.005	0.001
	$j=8$	-.082	0.033	0.026	-.009

space-time partial autocorrelation functions (shown in Table 7b), the distinct cutoff at time lag $j = 2$ (for each h) indicates that the time order is $r = 2$.

Suppose in equation (6.4), the $\beta_{11}^{22} = 0.7$ value is replaced by $\beta_{11}^{22} = 0$ but that all other parameter values are unchanged. The resulting sample space-time autocorrelation functions are provided in Table 8. Notice that the $\rho_1(j) = 0$ (for each j) as expected from equation (5.7) since $m = n = h = 1 < \max(\boldsymbol{\xi}, \boldsymbol{\mu})$. The same patterns in the autocorrelation and partial autocorrelation functions prevail and indicate the time order is $j = 2$ and the maximum spatial order is $h = 2$, as before.

Table 8: Sample space-time autocorrelation functions for $\{z^2(t)\}$: Example $3'$.

	$\rho_h(j)$	Spatial lag $h=0$	$h=1$	$h=2$	$h=3$
	$j=1$	-.039	-.036	-.024	-.010
Time	$j=2$	**0.490**	0.072	**0.181**	0.024
lag	$j=3$	-.038	-.035	-.027	-.011
	$j=4$	**0.211**	0.047	**0.136**	0.017
	$j=5$	-.039	-.030	-.028	-.007
	$j=6$	**0.153**	0.042	**0.075**	0.013
	$j=7$	-.029	-.025	-.021	-.011
	$j=8$	0.087	0.029	0.059	0.006

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