Advances in Methodology, Data Analysis, and Statistics
Anuška Ferligoj (Editor), Metodološki zvezki, 14
Ljubljana: FDV, 1998

Some Inference Results for the Exponential Autoregressive Process

Lynne Billard[1]

Abstract

After briefly describing the exponential autoregressive model of order p, EAR(p), introduced by Gaver and Lewis (1980) and Lawrance and Lewis (1980), we give conditional least squares estimators of the model parameters. Then, we obtain two predictors for the model for optimality with respect to the squared error loss function and the absolute loss function, respectively, and compare their relative merits.

1 Introduction

Standard Gaussian time series models have been studied extensively and are well developed for use in a variety of settings (see, for example, Box and Jenkins, 1976). In particular, the autoregressive model of order p, AR(p), is given by

$$Y_t = \phi_1 Y_{t-1} + ... + \phi_p Y_{t-p} + \epsilon_t \tag{1.1}$$

where $\{Y_t\}$ is a sequence of observations at time $t = 1, ..., n$, and $\{\epsilon_t\}$ is the error sequence, or white noise process, of standard normal $N(0, \sigma^2)$ variates. Thus, the marginal distribution of the observations Y_t is also that of a normal distribution. There are many situations in practice however in which the observations Y_t are not normally distributed. Specifically, we wish to consider the case in which the marginal distribution of the observations follows an exponential distribution. Direct analogues of (1.1) for exponentially distributed $\{Y_t\}$ are not feasible. However, there is rich class of models which address this question. We will focus herein on the basic class of exponential models developed by Gaver and Lewis (1980) and Lawrance and Lewis (1980); see Section 2.

In Section 3, we review briefly work on estimating the model parameters, and in Section 4, we consider prediction for the basic model.

[1] Department of Statistics, University of Georgia, Athens, GA 30602-1952, USA

2 The EAR(p) model

Following Gaver and Lewis (1980) and Lawrance and Lewis (1980), we define the exponential autoregressive model of order p, EAR(p), by

$$Y_t = \left\{ \begin{array}{ll} \alpha_1 Y_{t-1} & \text{with probability } a_1 \\ \alpha_2 Y_{t-2} & \text{with probability } a_2 \\ \vdots & \\ \alpha_p Y_{t-p} & \text{with probability } a_p \end{array} \right\} + \epsilon_t, \tag{2.1}$$

where $0 < \alpha_r < 1, r = 1, ..., p$, and where the probabilities can be written as functions of the parameters $\{\alpha_j, j = 1, ..., p\}$ as

$$\begin{aligned} a_1 &= 1 - \alpha_2, \\ a_r &= \prod_{j=2}^{r} \alpha_j (1 - \alpha_{j-1}), \quad r = 2, ..., p-1, \\ a_p &= \prod_{j=2}^{p} \alpha_j. \end{aligned} \tag{2.2}$$

We are given that the marginal distribution of Y_t is an exponential distribution with mean μ, E_t say. Then, by using the Laplace transforms of Y_t and ϵ_t, Gaver and Lewis (1980) showed that, for the EAR(1) model when $p = 1$,

$$\epsilon_t = \left\{ \begin{array}{ll} 0 & \text{with probability } \alpha_1 \equiv \alpha, \\ E_t & \text{with probability } 1 - \alpha. \end{array} \right. \tag{2.3}$$

For general p, Lawrance and Lewis (1980) showed that the distribution of the error terms ϵ_t is a mixture of a zero discrete component and p different exponential distributions whose means are a function of the model parameters μ and $\alpha_r, r = 1, ..., p$. For example, for the EAR(2) model, we have

$$\epsilon_t = \left\{ \begin{array}{ll} 0 & \text{with probability } \pi_0, \\ E_t & \text{with probability } \pi_1, \\ E_t^* & \text{with probability } \pi_2 \end{array} \right. \tag{2.4}$$

where E_t^* is an exponential variate with mean $\mu^* = \mu\alpha_2(1 + \alpha_1 - \alpha_2)$, and where

$$\pi_0 = \alpha_1/\lambda, \quad \pi_1 = (1 - \alpha_1)(1 - \alpha_2)/(1 - \alpha_2\lambda),$$

$$\pi_2 = [\alpha_1 - \alpha_1/\lambda - \alpha_2(\alpha_1 - \alpha_2)]/(1 - \alpha_2\lambda),$$

with $\lambda = 1 + \alpha_1 - \alpha_2$.

While it is clear that the structures mathematically of the Gaussian autoregressive model of (1.1) and the exponential autoregressive model of (2.1) are different, the properties of these two models have many similarities. For example, the autocorrelation functions have the same format. For the EAR(1) model, the autocorrelation function at lag k, ρ_k, is $\rho_k = \alpha^k$ which compares with AR(1)'s $\rho_k = \phi^k$, $k = 0, \pm 1, ...$ For the EAR(2) model, ρ_k satisfies the Yule-Walker equations

$$\rho_k = \alpha_1(1 - \alpha_2)\rho_{k-1} + \alpha_2^2\rho_{k-2}, \quad k = \pm 1, 2 \pm 2, ..., \tag{2.5}$$

and $\rho_0 = 1$, which is analogous to the AR(2) Yule-Walker equations

$$\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2}, \quad k \pm 1, \pm 2, ... \ .$$

From (2.5), it follows that when $0 < \rho_1 \leq 1/2$, we have $\rho_1^2 < \rho_2 < 1$, and when $1/2 \geq \rho_1 < 1$, we have $\rho_1^2 < [\rho_2 < 2\rho_1 - 1 - (1 - 1/\rho_1)^2]$ thus restricting the possible range of (ρ_1, ρ_2) values for the EAR(p) model. The parameters α_1 and α_2 can be expressed in terms of the autocorrelations as

$$\alpha_1 = \rho_1[1 + (\rho_2 - \rho_1^2)/(1 - \rho_1^2)]^{1/2}, \quad \alpha_2 = [(\rho_2 - \rho_1^2)/(1 - \rho_1^2)]^{1/2}. \tag{2.6}$$

In general, the Yule-Walker equations for the EAR(p) model are

$$\rho_k = \alpha_1 a_1 \rho_{k-1} + ... + \alpha_p a_p \rho_{k-p}, \quad k = \pm 1, \pm 2, ... \ . \tag{2.7}$$

Lawrance and Lewis (1980) also showed that the regression of Y_t on all p previous $\{Y_{t-r}, r = 1, ..., p\}$ is linear on these p $\{Y_{t-r}\}$, that is,

$$E(Y_t|Y_{t-1}, ..., Y_{t-p}) = \sum_{r=1}^{p} \alpha_r a_r Y_{t-r} + \mu(1 - \sum_{r=1}^{p} \alpha_r a_r), \tag{2.8}$$

but that the regression of Y_t on one of, not all, $\{Y_{t-r}, r = 1, ..., p\}$ is not linear.

3 Parameter estimation

Because of discontinuities in the likelihood function, the general method of Billingsley (1961) for obtaining maximum likelihood estimators cannot be applied to the EAR(p) process. However, Billard and Mohamed (1991) obtained conditional least squares estimators (CLS) of the parameters, and studied some properties of these estimators.

The CLS estimators $\hat{\alpha}_r$ of α_r, $r = 1, ..., p$, and $\hat{\mu}$ of μ, are those estimators which minimize

$$Q_n = \sum_{t=p+1}^{n} [Y_t - E(Y_t|Y_{t-1}, ..., Y_{t-p})]^2 \tag{3.1}$$

for a given set of observations $\{Y_t, t = 1, ..., n\}$, where $E(Y_t|y_{t-1}, ..., Y_{t-p})$ was given in (2.8). Hence, we can show that the $\hat{\alpha}_r$, $r = 1, ..., p$, are found from solving

$$C_{r0} = \hat{A}_1 C_{r1} + ... + \hat{A}_p C_{rp}, \quad r = 1, ..., p, \tag{3.2}$$

where

$$C_{rs} = (n-p)^{-1}[\sum_{t=p+1}^{n} Y_{t-r}Y_{t-s} - (n-p)^{-1}(\sum_{t=p+1}^{n} Y_{t-r})(\sum_{t=p+1}^{n} Y_{t-s})], \tag{3.3}$$

for $r, s = 1, ..., p$, and where, for $r = 1, ..., p$,

$$\hat{A}_r = \hat{\alpha}_r a_r \tag{3.4}$$

is the CLS estimator of $A_r = \alpha_r a_r$; and that the CLS estimator of μ is

$$\hat{\mu} = [\sum_{t=p+1}^{n} Y_t - \sum_{r=1}^{p} \hat{A}_r \sum_{t=p+1}^{n} Y_{t-r}]/[(n-p)(1 - \sum_{r=1}^{p} \hat{A}_r)]. \tag{3.5}$$

It is noted that in the particular case $p = 1$, the sequencing of runs down allows α to be determined directly, though it is still necessary to calculate $\hat{\alpha}$ to obtain $\hat{\mu}$ from (3.5).

Yule-Walker estimators $(\tilde{\alpha}_r, \tilde{\mu})$ of the parameters (α_r, μ) were suggested by Lawrance and Lewis (1980) as those estimators satisfying

$$c_r = \tilde{A}_1 c_{r-1} + ... + \tilde{A}_p c_{r-p}, \quad r = 1, ..., p, \tag{3.6}$$

where $\tilde{A}_r = \tilde{\alpha}_r a_r$ and where

$$c_r = n^{-1} \sum_{t=1}^{n-r} (Y_t - \bar{Y})(Y_{t+r} - \bar{Y}); \tag{3.7}$$

and

$$\tilde{\mu} = \bar{Y} = n^{-1} \sum_{t=1}^{n} Y_t. \tag{3.8}$$

Billard and Mohamed (1991) obtained results for the asymptotic distributions for the CLS estimators, as summarized in the following theorem.

Theorem:

Under certain regularity conditions, the CLS estimators $\hat{\boldsymbol{\beta}} = (\hat{\mu}, \alpha_1, ..., a_p)'$ of $\boldsymbol{\beta} = (\mu, \alpha_1, ..., \alpha_p)'$ are strongly consistent and asymptotically normally distributed as $n^{1/2}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \sim N(\mathbf{0}, \boldsymbol{C})$ where

$$\boldsymbol{C} = Var(\epsilon_t)\boldsymbol{V}^{-1}$$

where $\boldsymbol{V}$ is a $(p+1)(p+1)$ matrix with elements

$$V_{rs} = E(\frac{\partial g}{\partial \beta_r}, \frac{\partial g}{\partial \beta_s}), \quad r, s = 1, ..., p+1,$$

where $g \equiv E(Y_t|Y_{t-1}, ..., Y_{t-p})$ given in (2.8).

Corollary:

If we write $\hat{\boldsymbol{\beta}} = (\hat{\mu}, \hat{\boldsymbol{\alpha}})$, then $\hat{\mu}$ and $\hat{\boldsymbol{\alpha}}$ are asymptotically independent and normally distributed according to

$$n^{1/2}(\hat{\mu} - \mu) \sim N(0, Var\epsilon_t V_{11}^{-1}),$$

and

$$n^{1/2}(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}) \sim N(\mathbf{0}, Var\epsilon_t \boldsymbol{V}_{22}^{-1}),$$

respectively, where we write

$$\boldsymbol{V} = \begin{pmatrix} V_{11} & \boldsymbol{V}_{12} \\ \boldsymbol{V}_{21} & \boldsymbol{V}_{22} \end{pmatrix}$$

where V_{11} is the 1×1 scalar

$$V_{11} = E(\frac{\partial g}{\partial \mu})^2,$$

and $\boldsymbol{V}_{22}$ is the $p \times p$ matrix with elements

$$(V_{22})_{ij} = E(\frac{\partial g}{\partial \alpha_i}\frac{\partial g}{\partial \alpha_j}), \quad i, j = 1, ..., p.$$

While Lawrance and Lewis (1980) did not derive properties of the Yule-Walker estimators, it is clear from (3.3) and (3.7) that they have the same asymptotic distribution as do the CLS estimators. Based on simulation studies (for $p = 1$) when the sample sizes are small however, Billard and Mohamed (1991) showed that the CLS estimators consistently outperform the Yule-Walker estimators as judged by criteria of bias and mean square error results. Furthermore, the CLS estimators tended to attain their asymptotic values by $n = 50$, whereas the Yule-Walker estimators typically still had not reached the asymptotic values by $n = 100$, for larger values of α.

4 Predictors

Let us now consider the question of predicting a future observation Y_{k+h} given $\{Y_t, t = 1, ..., n\}$, denoted by $f(n, h)$ say. We derive two predictors, viz.,

$$f^{(1)}(n, h) = E(Y_{n+h}|Y_n, ..., Y_1) \tag{4.1}$$

which is the "best" predictor when optimality is with respect to the squared loss function, and

$$f_r^{(2)}(n, h) = M_r \tag{4.2}$$

with M_r, for $0 \leq r \leq 1$, satisfying

$$r = \int_{-\infty}^{M_r} \alpha F(y|Y_n, ..., Y_1)$$

with $F(\cdot|\cdot)$ denoting the conditional distribution function, which for $r = 1/2$ is the "best" predictor when optimality is taken with respect to the absolute loss function. We will assess the merits of each predictor by obtaining both the prediction mean square error and the prediction mean absolute error.

When the observations are normally distributed, both $f^{(1)}(n,h)$ and $f_r^{(2)}(n,h)$ with $r = 1/2$ lead to the same predictors. When the $\{Y_t\}$ are nonnormally distributed, there are no theoretical grounds for always choosing $f^{(1)}(n,h)$, nor for selecting $r = 1/2$ in $f_r^{(2)}(n,h)$. Therefore, we derive both predictors herein for the exponential process and compare them. We demonstrate the methodology for the EAR(1) case; predictors for the general EAR(p) model follow analogously.

The Predictor $f^{(1)}(n,h)$

For the EAR(1) process, Gaver and Lewis (1980) showed that

$$Y_{t+h} = \alpha^h Y_t + \alpha^{h-1}\epsilon_{t+1} + ... + \epsilon_{t+h}. \tag{4.3}$$

Since the general autoregressive process of order one, including the particular exponential autoregressive model, is a Markov process of order one, we have

$$f^{(1)}(n,h) = E(Y_{n+h}|Y_n, ..., Y_1) = E(Y_{n+h}|Y_n).$$

Therefore, from (4.3), we can obtain

$$f^{(1)}(n,h) = \alpha^h Y_n + \mu)(1-\alpha^h), \quad h = 1, 2, ... \ . \tag{4.4}$$

The prediction error of $f^{(1)}(n,h)$ is

$$e^{(1)}(n,h) = Y_{n+h} - \alpha^h Y_n - \mu(1-\alpha^h), \quad h = 1, 2, ... \ . \tag{4.5}$$

Clearly,

$$E[e^{(1)}(n,h)|Y_n] = 0 = E[e^{(1)}(n,h)]. \tag{4.6}$$

That is, $f^{(1)}(n,h)$ is an unbiased predictor of Y_{n+h}.

To derive the mean square error of $f^{(1)}(n,h)$, we first obtain the conditional mean square error given Y_n, $E_n[e^{(1)}(n,h)]^2$ say. Thus, we can show that

$$\begin{aligned} E_n[e^{(1)}(n,h)]^2 &= Var(Y_{n+h}|Y_n) \\ &= \mu^2(1-\alpha^{2h}). \end{aligned}$$

Hence, the unconditional mean square error of the predictor $f^{(1)}(n,h)$ is

$$E[e^{(1)}(n,y)]^2 = \mu^2(1-\alpha^{2h}), h = 1,2,\ldots\ . \tag{4.7}$$

Likewise, for the mean absolute error of $f^{(1)}(n,h)$, we first derive the conditional mean absolute error given Y_n, $E_n|e^{(1)}(n,h)|$. We can show

$$E_n|e^{(1)}(n,h)| = E|\xi - d|$$

where $d = \mu(1-\alpha^h)$, and

$$\xi = \alpha^{h-1}\epsilon_{n+1} + \alpha^{h-2}\epsilon_{n+2} + \ldots + \epsilon_{n+h}. \tag{4.8}$$

From (2.3), we can write

$$\begin{aligned} E_n|e^{(1)}(n,h)| &= \alpha^h E|-d| + \alpha^{h-1}(1-\alpha)\sum_{i=1}^{h} E|\alpha^{h-1}\epsilon_{n+1} - d| \\ &+ \alpha^{h-2}(1-\alpha)^2 \sum_{\substack{i=1 \\ i\neq j}}^{h}\sum_{j=1}^{h} E|\alpha^{h-i}\epsilon_{n+i} + \alpha^{h-j}\epsilon_{n+j} - d| \\ &+ \cdots \\ &+ (1-\alpha)^h E|\alpha^{h-1}\epsilon_{n+1} + \ldots + \epsilon_{n+h} - d|. \end{aligned} \tag{4.9}$$

Thus, we can evaluate the expectation in (4.9), though it is algebraically tedious. When $h = 1$, we easily have

$$E_n|e^{(1)}(n,1)| = \alpha d + (1-\alpha)[d - \mu + 2\mu\exp(-d/\mu)].$$

Hence, the unconditional mean absolute error for the predictor $f^{(1)}(n,1)$ is

$$E|e^{(1)}(n,1)| = 2\mu(1-\alpha)\exp[-(1-\alpha)]. \tag{4.10}$$

The Predictor $f_r^{(2)}(n,h)$

From (4.2), it is clear that to obtain the optimal absolute loss function predictor $f_r^{(2)}(n,h)$ we first need to derive the conditional distribution $F(y_{n+h}|Y_n,\ldots,Y_1)$. When $h = 1$, we have the conditional density function $f(y_{n+1}|Y_n)$ from Gaver and Lewis (1980). Hence, using their result to derive $f_r^{(2)}(n,1)$ first for $h = 1$, we can then use induction to derive the predictor for general h. Thus, we can show that

$$f_r^{(2)}(n,h) = \begin{cases} \alpha^h Y_n, & r \leq \alpha, \\ \alpha^h Y_n + \mu(1-\alpha^h)(1-\alpha)^{-1}ln[(1-\alpha)/(1-r)], & r > \alpha, \end{cases} \tag{4.11}$$

for $h = 1,2,\ldots$. Notice from (2.3) and (4.11) that the structure of $f_r^{(2)}(n,h)$ is similar to the structure underlying the model itself.

The prediction error of $f_r^{(2)}(n,h)$ is, for $h = 1, 2, ...,$

$$e_r^{(2)}(n,h) = \begin{cases} Y_{n+h} - \alpha^h Y_n, & r \leq \alpha, \\ Y_{n+h} - \alpha^h Y_n - \mu(1-\alpha^h)(1-\alpha)^{-1} ln[(1-\alpha)/(1-r)], & r > \alpha, \end{cases} \tag{4.12}$$

which can be written as

$$e_r^{(2)}(n,h) = \begin{cases} e^{(1)}(n,h) + c_1, & r \leq \alpha, \\ e^{(1)}(n,h) + c_2, & r > \alpha, \end{cases} \tag{4.13}$$

where

$$c_1 = \mu(1-\alpha^h), \quad c_2 = \mu(1-\alpha^h) - \mu(1-\alpha^h)(1-\alpha)^{-1} ln[(1-\alpha)/(1-r)].$$

Substituting from (4.6) into (4.13), we have that

$$E[e_r^{(2)}(n,h)] = \begin{cases} c_1, & r \leq \alpha, \\ c_2, & r > \alpha. \end{cases} \tag{4.14}$$

Hence, it follows that for a given α, $0 < \alpha < 1$, the predictor $f_r^{(2)}(n,h)$ is positively biased for r in $0 < r < r^*$, is unbiased for $r = r^*$, and is negatively biased for r in $r^* < r < 1$, where

$$r^* = 1 - (1-\alpha)\exp[-(1-\alpha)]. \tag{4.15}$$

Note that $r^* > \alpha$. Thus, the intuitive choice $r^* = 1/2$ does not in fact give the unbiased predictor; see also Table 1.

The mean square error of this predictor follows according to

$$E[e_r^{(2)}(n,h)]^2 = \begin{cases} \mu^2(1-\alpha^{2h}) + c_1^2, & r \leq \alpha, \\ \mu^2(1-\alpha^{2h}) + c_2^2, & r > \alpha. \end{cases} \tag{4.16}$$

Note that $E[e_r^{(2)}(n,h)]^2$ is minimized at $r = r^*$ and that

$$E[e_{r^*}^{(2)}(n,h)]^2 = E[e^{(1)}(n,h)]^2.$$

To derive the mean absolute error of $f_r^{(2)}(n,h)$, as before, we first derive the conditional mean absolute error given Y_n, viz.,

$$E_n|e_r^{(2)}(n,h)| = \begin{cases} E|\xi|, & r \leq \alpha, \\ E|\xi - d'|, & r > \alpha, \end{cases}$$

where ξ is given in (4.8) and

$$d' = \mu(1-\alpha^h)(1-\alpha)^{-1} ln[(1-\alpha)/(1-r)].$$

Again, this is tedious algebraically but otherwise straightforward to derive. In particular, when $h = 1$, we can then show that the unconditional mean absolute error is

$$E|e_r^{(2)}(n,1)| = \begin{cases} \mu(1-\alpha), & r \leq \alpha, \\ \mu\{1+\alpha-2r+ln[(1-\alpha)/(1-r)]\}, & r > \alpha. \end{cases} \tag{4.17}$$

The Two Predictors Compared

The predictor $f^{(1)}(n,h)$ is unbiased but $f_r^{(2)}(n,h)$ is not except at $r = r^*$. The bias for $\alpha = 0.1(0.2)0.9$ and $r = 0.1(0.2)0.9$ of $f_r^{(2)}(n,h)$ is shown in Table 1, for the case that $\mu = 1$. Also given is the value of r^* for given α. Thus, we observe that the bias is a concave downward function of α.

In Table 2, we compare the mean square errors of the two predictors for the one step ahead predictor $h = 1$, taking $\alpha = 0.1(0.2)0.9$, $r = 0.1(0.2)0.9$, and $\mu = 1$. We observe that as α increases, the prediction mean square error decreases. It also follows from (4.7) and (4.16) that the corresponding predictor mean square errors are related as

$$E[e^{(1)}(n,h)]^2 \leq E[e_r^{(2)}(n,h)]^2$$

with equality at $r = r^*$. The numerical values of Table 2 demonstrate this result. That is, according to the mean square error criterion, the predictor $f^{(1)}(n,h)$ is always to be preferred over the predictor $f^{(2)}(n,h)$.

In contrast, when judged by the mean absolute error criterion, the predictor $f_r^{(2)}(n,h)$ is generally preferred. In Table 3, the mean absolute errors for each predictor at $h = 1$ are given for $\alpha = 0.1(0.2)0.9$, $r = 0.1(0.2)0.9$, and $\mu = 1$. Again, these errors decrease as α increases. Also, though not true for all α and r values, it is generally the case that these errors are smaller for $f_r^{(2)}(n,1)$ than for $f^{(1)}(n,1)$. Further, unlike the mean square errors for $f_r^{(2)}(n,1)$ which were minimized at $r = r^*$, the mean absolute errors for $f_r^{(2)}(n,1)$ are minimized at $r = 1/2$.

We can show from (4.10) and (4.17) that at $r = 1/2$,

$$E|e^{(1)}(n,1)| > E|e_{1/2}^{(2)}(n,1)|$$

suggesting a preference for $f_{1/2}^{(2)}(n,1)$. Comparing the entries in Tables 2 and 3 for $r = 0.5$, we see that as α increases the gain by selecting $f_{1/2}^{(2)}(n,1)$ over $f^{(1)}(n,1)$ according to the mean absolute loss criteria, exceeds the loss by selecting $f^{(1)}(n,1)$ over $f_{1/2}^{(2)}(n,1)$ according to the mean square error criterian. This suggests a preference for the prediction $f_{1/2}^{(2)}(n,h)$ in these cases.

Finally, when $r = r^*$, both predictors are equivalent regardless of whether the predictor is selected according to the mean square error or mean absolute error criteria. The important caveat here is that r^* is a function of α and does not necessarily equal the intuitive choice that $r = 1/2$.

More complete details can be found in Mohamed and Billard (1990).

TABLE 1

Bias of $f_4^{(2)}(n,1)$ in EAR (1) model, $\mu = 1$

$\alpha\backslash r$	.1	.3	.5	.7	.9	r^*
.1	0.90	0.64	0.31	-0.19	-1.29	.634
.3	0.70	0.70	0.36	-0.14	-1.24	.652
.5	0.50	0.50	0.50	-0.01	-1.10	.697
.7	0.30	0.30	0.30	0.30	-0.79	.778
.9	0.10	0.10	0.10	0.10	0.10	.910

TABLE 2

Mean square errors of $f^{(1)}(n,1)$ and $f_r^{(2)}(n,1)$ in EAR(1) model, $\mu = 1$

	$E[e^{(1)}(n,1)]^2$	$E[e_r^{(2)}(n,1)]^2$				
$\alpha\backslash r$		.1	.3	.5	.7	.9
.1	0.99	1.80	1.16	1.09	1.03	2.65
.3	0.91	1.40	1.40	1.04	0.93	2.45
.5	0.75	1.00	1.00	1.00	0.75	1.96
.7	0.51	0.60	0.60	0.60	0.60	1.13
.9	0.19	0.20	0.20	0.20	0.20	0.20

TABLE 3

Mean absolute errors of $f^{(1)}(n,1)$ and $f_r^{(2)}(n,1)$ in EAR(1) model, $\mu = 1$

	$E\|e^{(1)}(n,1)\|$	$E\|e_r^{(2)}(n,1)\|$				
$\alpha\backslash r$		.1	.3	.5	.7	.9
.1	0.73	0.90	0.75	0.69	0.80	1.50
.3	0.70	0.70	0.70	0.64	0.75	1.45
.5	0.61	0.50	0.50	0.50	0.61	1.31
.7	0.44	0.30	0.30	0.30	0.30	1.00
.9	0.18	0.10	0.10	0.10	0.10	0.10

References

[1] Billard L. and Mohamed F.Y. (1991): Estimation of the parameters of an EAR(p) process. *Journal of Time Series Analysis*, **12**, 179-192.

[2] Billingsley P. (1961): *Statistical Inference for Markov Processes.* University of Chicago Press.

[3] Box G.E.P. and Jenkins G.Y. (1976): *Time Series Analysis: Forecasting and Control.* Holden Day.

[4] Gaver D.P. and Lewis P.A.W. (1980): First order autoregressive gamma sequences and point processes. *Advances in Applied Probability*, **12**, 727-745.

[5] Lawrance A.J. and Lewis P.A.W. (1980): The exponential autoregressive moving average EARMA(p, q) process. *Journal of Royal Statistical Society B*, **42**, 150-161.

[6] Mohamed R.Y. and Billard L. (1990): Prediction for the EAR(1) and EAR(2) processes. Technical Report.