Contributions to Methodology and Statistics
A. Ferligoj and A. Kramberger (Editors)
Metodološki zvezki, 10, Ljubljana: FDV 1995

Asymptotics of Dynamics for Control in Society

Vesna Omladič[1]

Abstract

In the discussed model of hierarchically ordered societies the macro structures of a society dynamically interact. This model helps to interpret some social and political phenomena such as coalitions, their hierarchical levels and the degrees of anarchy on a certain hierarchical level. The stability of the system is certain in the constant parameter case. Asymptotic behaviour of this kind of models is applied to the analysis of creating coalitions of two, three and more political structures. In the case of additive resources the problem is translated into a social choice problem with measurable value functions.

Keywords: Social power; Social control; Social choice; Hierarchy; Dynamical systems; Positive matrices.

1 Introduction

In his path breaking paper Allen (1992) proposes a model for the dynamics of political power and control in a social system. In his paper various socio-political phenomena such as coalitions, alliances, anarchy, and revolutions are introduced. However, there is no good treatment of hierarchical order in political structure. This partial imperfection of the model is improved by Omladič and Omladič (1994). It is a surprising feature of this model that the equilibria always exist and that asymptotically the system always tends to it.

Let us present here in short the Allen's macro viewpoint: There is a finite number of *structures* in the society, i.e. its control holding, power wielding categories, which possess relative independence of each other. These categories exercise power over other structures and subjects of the society through their substructures and various organisations. Here are the somewhat simplified Allen's axioms on socio-political power and control that are clearly independent of particular theories:

Axiom 1. Each of the structures attempts to exert power to control the others.

Axiom 2. Each of the structures accedes to the others a certain fraction of its complete control.

[1]Faculty of Social Sciences, University of Ljubljana, P.O. Box 47, 61 109 Ljubljana, Slovenia

Axiom 3. Each structure acts in such a way to align the fraction of control that another structure has over an object according to an intrinsic control column, which may vary from structure to structure.

Denote by n the number of identifiable structures and let the structures be denoted by $S_i, i = 1, 2, \ldots, n$. We will fix a structure (or an object) over which the structures exert their power. For any indices i, j let p_{ij} denote the fraction of power that structure S_j accedes to structure S_i out of what it has over the fixed object. This matrix of power fractions is called *power profile.* It follows from this definition that

$$\sum_{i=1}^{n} p_{ij} = 1, \tag{1}$$

i.e. the $n \times n$ power profile matrix $P = (p_{ij})$ is column stochastic. This condition means that all control that a certain structure S_j has over the fixed object gets allocated. For any structure we further define x_j as the fraction of control that structure S_j has over the object under consideration. We again must have that

$$\sum_{j=1}^{n} x_j = 1, \tag{2}$$

in order to preserve the total amount of control. These fractions form the power fraction or control fraction vector $x = (x_j)$. Each structure has certain *resources* or *powers* at its disposal in order to achieve its goals. Denote by r_j the primary or direct powers that structure S_j can exert to achieve its aims over the object. In order to write down Allen's model in matrix notation we introduce the resource matrix $R = \operatorname{diag}(r_1, r_2, \ldots, r_n)$, notation $\mathbf{e}$ for the n-tuple made of 1's and the resource vector $r = R\mathbf{e}$. Moreover, denote by $\langle x, y \rangle$ the usual scalar product between the real n-tuples $x, y \in \mathbb{R}^n$. The model can now be written as

$$\dot{x} = (PR - \langle r, x \rangle)x, \tag{3}$$

while conditions (1) and (2) become

$$P^{\mathrm{tr}}\mathbf{e} = \mathbf{e} \quad \text{and} \quad \langle x, \mathbf{e} \rangle = 1, \tag{4}$$

where P^{tr} means the transposed matrix of matrix P.

A complete discussion of this model, including its comparison to some other social models, was given by Allen (1992). Nevertheless, a brief explanation of the terms appearing in equation (3) will not be amiss. It relates the rates of growth (or decrease) of fractions of control on the left-hand side with the difference between the gained power and the acceded one on the right-hand siede. While the vector of gained powers is equal to PRx and has a clear interpretation, the acceded one must be proportional to the vector of fractions of control. Let us denote this proportion by c to see that the right-hand side of (3) must be of the form $PRx - cx$. Now, since the growth of the fraction of control of some structure can only be gained at the expense of decrease of the fractions of control of some other structures, we must have that the sum of the growth-rates must be zero. This implies that $0 = \langle \mathbf{e}, \dot{x} \rangle = \langle \mathbf{e}, PRx \rangle - c\langle \mathbf{e}, x \rangle = \langle r, x \rangle - c$ which determines the coefficient c uniquely.

Hence, we search a solution $x(t)$ of the system of differential equations (3) satisfying a given initial condition $x(0)$. If we denote by $\mathcal{H}_1$ the hyper plane $\{x \in \mathbb{R}^n;\ \langle x, e\rangle = 1\}$, and by Δ the simplex, obtained as an intersection of the hyper plane $\mathcal{H}_1$ with the positive cone $\mathbb{R}^n_+$, then the second of the equations in (4) means geometrically that we are seeking for a solution of (3) lying in the hyper plane $\mathcal{H}_1$. But the intrinsic condition that the entries of x are non-negative, because they are fractions, actually requires x to remain in Δ. It was shown by Omladič and Omladič (1994) that if this condition is imposed on the initial vector $x(0)$ then automatically $x(t) \in \Delta$ for all $t \geq 0$.

The questions of existence, uniqueness and asymptotic stability of the equilibrium points of system (3) were treated by Allen (1992) and Omladič and Omladič (1994). We want to present here briefly these results and give some applications to the study of creating coalitions of political structures. The preliminary mathematical and socio-political definitions and interpretations will be given in Section 2, while Section 3 is reserved for the presentation of the main results from Allen (1992) and Omladič and and Omladič (1994). In Section 4 we apply the theory to the study of how two structures are creating coalitions, in Section 5 we consider forming the coalitions of three structures in a model with additive resources, while in Section 6 we give a general approach to this kind of problems.

2 Preliminaries

We shall write $x \geq 0$ for a vector $x \in \mathbb{R}^n$, if it belongs to the positive cone $\mathbb{R}^n_+$, i.e. if all of its entries are non-negative. If they are all strictly positive, we shall write $x > 0$. An analogous notation will be used for an $n \times n$ real matrix A. The spectral radius of A, i.e. the maximum of the absolute values of its (complex) eigenvalues, will be denoted by $\rho(A)$. We will only be interested in the real eigenvalues $\lambda \in \mathbb{R}$ and corresponding real eigenvectors $x \in \mathbb{R}^n$. If we can find for a given eigenvalue $\lambda \in \mathbb{R}$ a sequence of vectors $x_1, \ldots, x_m \in \mathbb{R}^n$, $x_m \neq 0$, such that

$$Ax_1 = \lambda x_1 + x_2, \quad \ldots, \quad Ax_{m-1} = \lambda x_{m-1} + x_m, \quad Ax_m = \lambda x_m,$$

we call x_1 a *root vector*, or equivalently *generalised eigenvector* of A at λ. All our root vectors will be taken at $\lambda = \rho(A)$. The length m of this sequence is uniquely determined by x_1 provided that $x_m \neq 0$ and will be called the *order of root vector* x_1. The mathematical part of the results presented in this section may be found in Gantmacher (1971), Schaefer (1975), Seneta (1981), and in Rothblum (1975), Zijm (1983), Schneider (1986).

A square non-negative matrix A is called *irreducible* if there is no permutation matrix Q such that

$$Q^{\mathrm{tr}}AQ = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}.$$

A matrix will be said *permutation matrix* if it is made of zeros and ones such that every column and every row contains exactly one entry equal to one. We say that $Q^{\mathrm{tr}}AQ$ is *permutationally similar* to A. The key result of the theory of non-negative matrices was obtained almost a hundred years ago by Perron and

Frobenius. It says that every $n \times n$ matrix A with non-negative entries has $\rho(A)$ as its eigenvalue and that there is an eigenvector x corresponding to this eigenvalue such that $x \geq 0$. Moreover, if A is irreducible, then $\rho(A) > 0$, and $x > 0$ is unique up to a multiplicative constant. We sometimes call $\rho(A)$ the *PF* eigenvalue and any corresponding non-negative eigenvector a *PF* eigenvector of A.

Motivated by this theorem and in view of our applications we shall call a square matrix A with non-negative entries *to be of PF type*, if

- $\rho(A) > 0$,

- there exists an eigenvector $x > 0$ with respect to eigenvalue $\lambda = \rho(A)$,

It is well-known that these matrices have the following canonical form. If a matrix A is of *PF* type then there is a permutation matrix Q and a block division such that

$$Q^{tr}AQ = \begin{bmatrix} A_0 & * & * & \cdots & * \\ 0 & A_1 & 0 & \cdots & 0 \\ 0 & 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_r \end{bmatrix}, \tag{5}$$

where r is the maximal number of linearly independent eigenvectors at the eigenvalue $\lambda = \rho(A)$, $\rho(A_i) = \rho(A)$ and A_i are irreducible for $i = 1, 2, \ldots, r$, stars are some non-zero blocks of appropriate dimension, and if the zeroth block exists, it holds that $\rho(A_0) < \rho(A)$.

Every non-negative matrix which is not nilpotent, i.e. having $\rho(A) > 0$, can be expressed in a canonical form with *PF* matrices on the block diagonal. More precisely: For every such matrix A there is a permutation matrix Q and a block division such that

$$Q^{tr}AQ = \begin{bmatrix} A_1 & * & * & \cdots & * & * \\ 0 & A_2 & * & \cdots & * & * \\ 0 & 0 & A_3 & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_k & * \\ 0 & 0 & 0 & \cdots & 0 & A_{k+1} \end{bmatrix}, \tag{6}$$

where k is the highest possible order of *PF* root vectors $\rho(A_l) = \rho(A)$, A_l are of *PF* type for $l = 1, 2, \cdots, k$, and if the $(k+1)$-st block exists, it holds that $\rho(A_{k+1}) < \rho(A)$.

The block form of A given in (6) is essentially unique. We can easily deduce from this canonical form a striking generalisation of the Perron-Frobenius theorem due to Rothblum. Namely, for any non-negative and non-nilpotent matrix there exist non-negative and non-zero root vectors forming a basis of the entire root space corresponding to *PF* eigenvalue. Each of the blocks A_l for $l = 1, 2, \ldots, k$ in (6) has a block representation as given in (5). The dimension of the root space equals the total number of irreducible blocks having the *PF* eigenvalue equal to $\rho(A)$ that appear after introducing (5) into (6).

It will turn out that the behaviour of the solution x of system (3) depends on the structure of matrix $A = PR$ which contains the power profile of the society under consideration, weighted by the actual primary powers that socio-political structures exercise over the fixed structure in question. In view of this application we will present some obvious interpretations of the above notions.

Let us first introduce the notion of subordination. We will call a structure S_j to be *immediately subordinated* to structure S_i if $p_{ij} > 0$, and *subordinated* to it if there is a sequence of indices $i = i_0, i_1, \ldots, i_m = j$ such that structure with index i_r is immediately subordinated to structure with index i_{r-1} for all $r = 1, 2, \ldots, m$. Therefore, immediate subordination means that structure S_j is acceding some of its primary power to structure S_i, while subordination means that this structure is acceding some of its powers to some structure which is acceding some of that to another, and so on, which is acceding some of it to structure S_i. We will say in accordance with [Oml] that a set of structures S is a *coalition* if all the structures that belong to S are sharing their powers with each other in some more or less intrinsic way, i.e. they are all subordinated to each other, and if the structures outside S may be subordinated to them, or may subordinate them, but none of them can do both. For any set of structures S, and in particular for any coalition, the according matrix A_S obtained from A by cutting out all the rows and columns with indices that do not belong to S will be called its *inner power matrix.* A coalition will be said to be *in power* if its inner power matrix has the same *PF* eigenvalue as the whole of A.

Recall the form (6) of matrix A. A structure S_j will be said to *belong to the l-th hierarchical level*, for $l = 1, 2, \ldots, k, k+1$, if index j belongs (after being permuted by permutation corresponding to matrix Q) to the l-th block of (6). Thus, the inner power matrix of the l-th hierarchical level is A_l. It is clear that if one structure belongs to a certain hierarchical level, then all the structures that are in coalition with it belong to the same level, so that we can talk about the *hierarchical level of a coalition.* It turns out that every coalition in power belongs to one of the first k blocks in (6).

The actual inner power relations between structures of a certain hierarchical level may be seen from the canonical form (5). The blocks $1, 2, \ldots, r$ represent all the coalitions in power. They do not co-operate and are subordinated to the coalitions on the zeroth block that are not in power. The top structures do not have enough power of their own, nevertheless, they become almost equally powerful as the non-cooperative coalitions in power that are subordinated to them, because they are gaining the power from them due to their hierarchical position. The number of coalitions in power on a certain level of hierarchy is called the *degree of anarchy* of this hierarchical level. If there is only one coalition in power on level l, say, we will term that there is *no anarchy* on this level.

3 Stability theorems

This section is devoted to the presentation of some of the main results of Allen (1992) and Omladič and Omladič (1994) on stability of the solution of (3). An explicit solution to this system may be written as

$$x(t) = \frac{e^{At}x(0)}{\langle \mathrm{e}, e^{At}x(0)\rangle}. \tag{7}$$

Let l be a fixed hierarchical level and write its inner power matrix according to (5)

$$A_l = \begin{bmatrix} C_0 & * & * & \cdots & * \\ 0 & C_1 & 0 & \cdots & 0 \\ 0 & 0 & C_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & C_r \end{bmatrix}, \tag{8}$$

where $C_1, C_2, \ldots, C_r$ are the inner power matrices of coalitions of this hierarchical level which are in power, while C_0 is the inner power matrix of the rest of the structures of this level which are not in power. Write any vector x with as many entries as there are structures on the l-th level according to this block division as

$$x = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_r \end{bmatrix}.$$

Since A_l is of *PF* type, all of its root vectors are of the first order and there are exactly r linearly independent ones. It turns out that one can choose root vectors in such a way that

$$x_1 = \begin{bmatrix} u_1 \\ v_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} u_2 \\ 0 \\ v_2 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad x_r = \begin{bmatrix} u_r \\ 0 \\ 0 \\ \vdots \\ v_r \end{bmatrix}, \tag{9}$$

where $u_j, v_j > 0$ for $j = 1, 2, \ldots, r$.

In the bigger matrix (6) we can find r root vectors of order l of the matrix

$$A = \begin{bmatrix} A_1 & * & * & \cdots & * & * \\ 0 & A_2 & * & \cdots & * & * \\ 0 & 0 & A_3 & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_k & * \\ 0 & 0 & 0 & \cdots & 0 & A_{k+1} \end{bmatrix},$$

using the r partial vectors given above. The root vectors can be chosen so that

$$y_1 = \begin{bmatrix} w_{11} \\ w_{21} \\ \vdots \\ w_{l-1\,1} \\ x_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad y_2 = \begin{bmatrix} w_{12} \\ w_{22} \\ \vdots \\ w_{l-1\,2} \\ x_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad y_r = \begin{bmatrix} w_{1r} \\ w_{2r} \\ \vdots \\ w_{l-1\,r} \\ x_r \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{10}$$

where $w_{pq} > 0$ for $p = 1, 2, \ldots, l-1$ and $q = 1, 2, \ldots, r$.

For an arbitrary hierarchical level l' let $P_{l'}$ denote the co-ordinate wise projection on the block of structures of the l'-th hierarchical level with the inner power matrix $A_{l'}$. Here, l' can be any index $l' = 1, 2, \ldots, k+1$. Thus, for the fixed hierarchical level $l' = l$ and for vectors in (10) it holds that $P_l y_i = x_i$ for $i = 1, 2, \ldots, r$. For any fraction-control vector $x \in \Delta$ we have that $P_l x$ represents the fractions of control given to the structures on the l-th hierarchical level. Similarly let P_S be the projection to the block corresponding to any set S of structures. We say that a fraction-control vector $x \in \Delta$ *gives immediate control to a structure* S_i if $x_i > 0$. We say that it *gives immediate control to a coalition* S if $P_S x \neq 0$. Further, we say that it *gives control to a coalition in power* if it gives immediate control to it or to a structure subordinated to it. In this case we call *its hierarchical level* to be the lowest level l, i.e. the greatest index l, such that it gives control to a coalition in power on the l-th hierarchical level. The following theorems were given in Omladič and Omladič (1994).

Theorem A. *The equilibrium exists and gives immediate control only to structures on the first hierarchical level. Let (8) be the inner power matrix of that level and let x_i and y_i with $l = 1$, normed so that they belong to Δ, for $i = 1, 2, \ldots, r$ be given respectively by (9) and (10). Then all the equilibrium points that are giving control to some coalition in power are exactly the convex combinations of these r points. In particular, the equilibrium is unique if and only if there is no anarchy on the first hierarchical level.*

Theorem B. *Assume that the control is given initially, i.e. by the initial condition $x(0)$, to a coalition in power. Then solution (7) of system (3) exists and so does the limit $\lim_{t\to\infty} x(t) = x$ which is equal to one of the equilibrium points. Thus, this limit gives control only to structures on the first hierarchical level. The limit is unique and independent of the initial condition if and only if there is no anarchy on the first hierarchical level.*

Theorem C. *Assume that the control is given initially, i.e. by the initial condition $x(0)$, to a coalition in power and let l be its hierarchical level. Then, there exist an equilibrium point u and constants $K, c > 0$ such that*

$$\|x(t) - u\| \le K t^{-1} \quad \text{if} \quad l > 1 \qquad \text{and} \qquad \|x(t) - u\| \le K e^{-ct} \quad \text{if} \quad l = 1.$$

In both cases $\|\,.\,\|$ *is any norm on* $\mathbb{R}^n$ *fixed in advance.*

4 Two-way coalitions: constant resources

Consider the simplest possible situation of any interest in which there are two structures S_1 and S_2 with resources r and s. Without loss of generality we may assume that $r \geq s$ since otherwise we could interchange the two structures. If the two structures do not want to co-operate, there is nothing to analyse: The only possible weighted power profile matrix is

$$A = \begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix}.$$

Proposition A.

1. *If* $r > s$*, there is no anarchy on the first hierarchical level which consists only of the first structure.*
2. *If* $r = s$*, the degree of anarchy on the first (and only) hierarchical level is 2.*

Indeed, if $r > s$, the *PF* eigenvalue is r and the corresponding eigenvector is

$$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

In this case the first structure is in power so that it gains all the control in the long run. In case that $r = s$, vector x is still a *PF* eigenvector, but we have another one

$$y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

In this case both structures are in power, and, because they do not co-operate, the degree of anarchy on the only hierarchical level is 2. Actually, the system is in the state of pure anarchy, since we only have two structures in it. The limiting point of the solution always exists, but may attain any possible vector of fractions from Δ.

A richer theory may be obtained when we allow the two structures to co-operate. We can view the columns of the power profile matrix

$$p = \begin{bmatrix} a \\ 1-a \end{bmatrix} \quad \text{respectively} \quad q = \begin{bmatrix} 1-b \\ b \end{bmatrix}$$

as strategies or policies that the two structures have in order to achieve their goals when creating the coalition. Here, a and b may be any real numbers from the interval $[0, 1]$. The weighted power profile matrix in this situation becomes

$$A = \begin{bmatrix} ra & s(1-b) \\ r(1-a) & sb \end{bmatrix}.$$

Proposition B.

1. *If $r > s$ and if each structure is maximising its asymptotical fraction of control, it is optimal for the first structure to choose $a = 1$. The second one cannot prevent the first one to gain all the control in the long run.*

2. *If $r = s$ all the possible $a, b \in [0, 1)$ lead to a non-anarchic distribution of control between the two structures.*

The *PF* eigenvalue ρ of A clearly satisfies the quadratic equation

$$\rho^2 - (ra + sb)\rho + rs(a + b - 1) = 0. \tag{11}$$

We can easily compute ρ from here in terms of a and b, since it must be the bigger of the two real solutions of this equation. Then, we can express the entries of the *PF* eigenvector x, defined by $(A - \rho)x = 0$ in terms of a and b. Then, we can maximise the asymptotical fraction of control of each of the two structures. Thus, S_1 wants the first entry of x to be as big as possible, and S_2 wants the second one to be as big as possible. Thus, their preferences are antagonistic.

We will do this now using a trick that will simplify the computations substantially. Starting from equation (11) we will not, at first, compute ρ in terms of a and b, but we will compute b in terms of a and ρ. What we get is

$$b = \frac{\rho^2 - ra\rho + rs(a-1)}{s(\rho - r)}.$$

Equation $(A - \rho)x = 0$ yields a system of linear equations in x_1 and x_2:

$$\begin{array}{rcrcl} (ra - \rho)x_1 & + & s(1-b)x_2 & = & 0 \\ r(1-a)x_1 & + & (sb - \rho)x_2 & = & 0. \end{array}$$

Let us now insert the above expression for b into it to obtain

$$\begin{array}{rcrcl} (ra - \rho)x_1 & + & \dfrac{-\rho^2 + (s + ra)\rho - rsa}{(\rho - r)}x_2 & = & 0 \\ r(1-a)x_1 & + & \dfrac{r(1-a)(\rho - s)}{(\rho - r)}x_2 & = & 0. \end{array}$$

Since the determinant of the system is zero, we can keep only one of the equations. We also have to recall the normalising condition $x_1 + x_2 = 1$ in order to get after a short, but straightforward computation:

$$x_1 = \frac{\rho - s}{r - s} \quad \text{and} \quad x_2 = \frac{r - \rho}{r - s}.$$

Now, note that x_1 and x_2 can be computed using these formulas only when $r > s$. Therefore, assume this condition and observe that these two numbers do not depend on a and b explicitely, but only through ρ as a solution of equation (11). If we find out that they are non-negative, we will know that they are the wanted equilibrium fractions of control. However, it is well-known that the *PF* eigenvalue of any non-negative matrix lies between the minimum and the maximum sum of its rows or columns. Since the sum of the first column of A is r and the sum of the second is s, it follows that $x_1, x_2 \geq 0$.

We have thus seen that in the case $r > s$ we must have that $r \geq \rho \geq s$ and it is in the interest of S_1 to make ρ as big as possible, while it is in the interest of S_2 to make ρ as small as possible. Thus, the antagonism between the two structures may easily be expressed through ρ. If there is no other criteria it seems that it is the best policy for the first structure to keep all the power to itself, thus pushing the second one out of power and out of control. In this case S_2 has no means to cope with the situtation. Whatever it does, it always remains on the second hierarchical level which is not in power. In a long run, all the control goes to S_1.

It remains to treat the case when $r = s$. In this case we clearly have $\rho = r$. Because the two structures are willing to co-operate, we suppose $a, b < 1$. It follows that the only *PF* eigenvector is given by

$$x_1 = \frac{1-b}{2-a-b} \quad \text{and} \quad x_2 = \frac{1-a}{2-a-b}.$$

Since the positions of the two players in this antagonistic game is symmetric they may decide to go for the Nash bargaining and split the control equally. Note that this condition forces $a = b$. This means that S_1 must accede the same portion of its primary power to S_2 as the other way arround. The final result in this case seems the same as in the non-cooperative possiblity. However, if they did not co-operate, the system would be in pure anarchy and the asymptotic result unclear. So, it seems better for the two structures to settle on one half of the control each.

5 Three-way coalitions: additive resources

In the previous section we considered the problem of forming coalitions in a model with constant resources. The problem of creating coalitions in this case was transformed into a very simple antagonistic game with two players. Coalitions of three and more structures could be studied in a similar way, but results would not differ substantially. In the kind of antagonistic games the players with different power would separate, coalitions could only be formed between structures of equal power and in this case the strongest structure or coalition would win.

It may also be interesting to treat a non-antagonistic situation in which the structures are actually gaining resources when entering a coalition. A possible model may be based on the following assumptions: Every structure S_i has its individual resources. However, for a structure that entered a coalition formed by more structures its resources equal the sum of the individual resources of all the members of the coalition. We also assume that the goal of the structures in question is to raise as much control as possible in a long run. According to the model presented in the previous section this means that each of them wants its entry in the limit of the solution of (3) to be as big as possible, and this implies that its entry in the *PF* eigenvector of A should be as big as possible. It is natural to refer to this kind of rules for creating coalitions as *model with additive resources.* These models are motivated by situtations where structures are forming a group voting through the weighted majority rule. In this case the individual resources of a structure may mean its weight in the voting procedure. After forming a coalition the total resources of any structure in the coalition become the sum of the voting weights of its members.

Let us consider a three-way model with additive resources. Assume that the three structures under consideration have individual resources equal to $r \geq s \geq t$. We will analyse the model using decision theory and the reader will be assumed familiar with elementary notions of this theory as presented in French (1986), say. We first pose the question whether the first structure is prepared to go into a coalition or not. If it stays alone, and if the other two stay alone as well, it will gain the maximal possible fraction of control, i.e. it will gain the whole control. However, if the other two decide for a coalition, matrix A of the model becomes

$$\begin{bmatrix} r & 0 & 0 \\ 0 & s & s \\ 0 & t & t \end{bmatrix}. \tag{12}$$

Now, if $r > s+t$ the *PF* eigenvalue of this matrix is r with eigenvector $x = (1,0,0)^{\mathrm{tr}}$ and the first structure still prefers to stay alone. If, on the other hand, $r < s+t$, then the *PF* eigenvalue of A becomes $s+t$ with normalised eigenvector

$$x = \begin{bmatrix} 0 & \frac{s}{s+t} & \frac{t}{s+t} \end{bmatrix}^{\mathrm{tr}}. \tag{13}$$

In this case the first structure loses the control completely.

Proposition C. *Assume that $r \geq s \geq t$ and that each structure is maximising its asymptotical fraction of control using the Wald's maximin return criterion when in a situtation of strict uncertainty.*

1. *If $r > s+t$ and it is optimal for the first structure not to accede any power to the others. The other two cannot prevent the first one to gain all the control in the long run.*

2. *If $r < s+t$ the coalition $\{S_2, S_3\}$ is the only one that is making both partners the most satisfied. It can gain all the power and all the control in the long run.*

Let us study the second case in somewhat more detail. Since there is no way in our model how to estimate the probability that the other two structures would form a coalition, the first structure finds itself in a situtation of strict uncertainty. Because staying alone results in no control and creating a coalition results in some control (as may be seen from (12) and (13) after permuting the structures and their resources accordingly), the structure will most certainly go for co-operation if it chooses the Wald's maximin return criterion. Now, what are the choices within this case? The coalition $\{S_1, S_2\}$ results in a fraction $\frac{r}{r+s}$ of control, coalition $\{S_1, S_3\}$ results in a fraction $\frac{r}{r+t}$, while overall coalition results in $\frac{r}{r+s+t}$ provided that the other partners are willing to co-operate in this way. As the goal of the structure is to gain the maximal possible control, it is clear from $r > s > t$ that S_1 ranks its possibilities for a coalition like this: 1. $\{S_1, S_3\}$, 2. $\{S_1, S_2\}$, 3. complete coalition, 4. no coalition.

Next, suppose that the other two possible partners have made an analogous analysis of the situation. Their considerations should then result in a ranking: 1. $\{S_2, S_3\}$, 2. $\{S_1, S_2\}$, 3. complete coalition, 4. no coalition for S_2; and 1. $\{S_2, S_3\}$, 2. $\{S_1, S_3\}$, 3. complete coalition, 4. no coalition for S_3. Can we guess the final result of these considerations? A short glance at the rankings shows that coalition $\{S_2, S_3\}$ is the only one that is making both partners the most satisfied.

In the next section we will propose an algorithm predicting the result in general case of n structures that will give the same answer. To conclude the section let us point out that this model could be improved in many ways to become more sophisticated. For instance, we could alow some partners within the coalition not to co-operate. This would simply mean that they are not exchanging mutually their individual powers. Thus, if in the above situation after the three structures have already decided for a complete coalition we want to compare the case of total co-operation with the possibility that the second two structures are non-cooperative, then we have to compare the *PF* eigenvectors and eigenvalues of matrices

$$A = \begin{bmatrix} r & r & r \\ s & s & s \\ t & t & t \end{bmatrix} \quad \text{and} \quad A' = \begin{bmatrix} r & r & r \\ s & s & 0 \\ t & 0 & t \end{bmatrix}.$$

Inserting explicit figures $r = 22$, $s = 15$, $t = 14$, into it, we obtain that the *PF* eigenvalue of A is $\rho = 51$ and the eigenvector is $r = (0.431, 0.294, 0.275)^{\text{tr}}$, while the same results for A' are $\rho = 43.8$ and the eigenvector is $r = (0.502, 0.262, 0.236)^{\text{tr}}$. This means that non-cooperation of the two structures decreases somewhat the power of the whole coalition measured by ρ, while inside the coalition structure S_1 is gaining over 15% of its control at the expense of the mutually non-cooperating partners. Let us point out that this example is motivated by the situation within the government coalition after the 1992 elections in the Republic of Slovenia.

6 General model with additive resources

We will now try to generalise the model with additive resources presented in previous section. Thus, we suppose that in addition to Allen's axioms the following axioms are satisfied.

Axiom 4. Every structure has its individual resources.

Axiom 5. The resources of any structure in a coalition equal the sum of the individual resources of its members, but it accedes all the gained power to the other members of the coalition.

Axiom 6. The goal of the structures is to attain maximal possible control in a long run.

We have thus obtained a model about which we will be hopefully able to tell more than about the general one introduced by Allen (1992) and Omladič and Omladič (1994). Actually, we would like to predict on the basis of these data what kind of coalitions will these structures create. We will denote the number of structures by n again. Denote by N the set of indices $\{1, 2, \ldots, n\}$. Any coalition can then be represented by a subset $I_{\mathcal{C}} \subset N$ so that for coalition $\mathcal{C}$ we have that

$$\mathcal{C} = \{S_i | \, i \in I_{\mathcal{C}}\}.$$

Let us try to see what are the *PF* eigenvector and eigenvalue of matrix A in case that coalition $\mathcal{C}$ has been formed. Assume that this coalition has k members and assume further for the sake of easier handling with computations that actually $\mathcal{C} = \{1, 2, \ldots, k\}$. Note that if this was not so, we could permute the indices to achieve this in order to get the formulas below.

Therefore, matrix A must be of the form

$$A = \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}, \quad \text{where } C = \begin{bmatrix} r_1 & r_1 & \cdots & r_1 \\ r_2 & r_2 & \cdots & r_2 \\ \vdots & \vdots & \ddots & \vdots \\ r_k & r_k & \cdots & r_k \end{bmatrix}$$

is the inner power matrix of the coalition $\mathcal{C}$ and D is the inner power matrix of the set of structures which do not belong to this coalition to be denoted by $\mathcal{D}$. Of course, each of the matrices C and D has a *PF* eigenvalue and corresponding eigenvector or even more of them. Since in our model the only possible way of acceding the power to some structure is to enter a coalition, we conclude that members of $\mathcal{C}$ do not co-operate in any way with the structures outside the coalition. This can be seen from the block diagonal structure of matrix A.

Let us compute the *PF* eigenvector and eigenvalue of matrix C. Note that $\mathbf{e}$ is a left *PF* eigenvector of C with corresponding eigenvalue equal to $\rho(C) = \sum_{i=1}^{k} r_i$. The same eigenvalue must yield the wanted right eigenvector as well. The easiest way to find it seems to go through an observation that C is diagonally similar to its transpose C^{tr}, similarity being performed by a matrix having r_i on the main diagonal. After a short computation we see that the vector

$$x = \begin{bmatrix} \frac{r_1}{\sum_{i=1}^{k} r_i} & \frac{r_2}{\sum_{i=1}^{k} r_i} & \cdots & \frac{r_k}{\sum_{i=1}^{k} r_i} \end{bmatrix}^{\text{tr}}$$

is the corresponding eigenvector of C. All the members of coalition can therefore find their fraction of control in a long run in this vector provided that their total power, i.e. $\rho(C)$, is greater than the total power of the rest of the structures, i.e. $\rho(D)$.

Now, what happens if their power is no greater than the power of the others? Actually, the possible members of the coalition have found themselves in a situation of strict uncertainty in this case. Namely, they have no idea of what kind of coalitions the others may consider or decide for. There are at least four well-known criteria to treat the kind of situation. For various reasons it is the Wald's pessimistic approach that seems the most appropriate for the situation, mainly because it appears to be the only among the considered ones to assure an easily obtainable preference relation that can be expressed with a measurable value function as will be seen in the sequel. And this is a very important and useful fact that will be needed in further analysis.

So, if we want to use the Wald's maximin return principle on the asymptotical fraction of control, we have to find the worst possible situation for the members of coalition. Using Perron-Frobenius theory it can be seen that D has the greatest possible *PF* eigenvalue exactly when all the structures from $\mathcal{D}$ form a coalition. In this case their *PF* eigenvalue clearly becomes $\rho(D)$ computed by the same formula from the individual power resources of the members of $\mathcal{D}$ as $\rho(C)$ above. Thus, $\rho(D)$ must be the sum of their individual resources. Hence, in the worst case the members of $\mathcal{C}$ have to compare their total power with the total power of their potential oponents. If they are stronger, then they have nothing to worry about, but, if they are weaker, then none of them will have any control in a long run, so that their pessimistically expected outcome will be zero. Finally, if they are equally powerful, the system is in anarchy of degree two on the only hierarchical level and the outcome is again unpredictible. However, since we have to go again for the Wald's pessimistic approach for the sake of consistency, there is a chance that members of $\mathcal{C}$ will get nothing in a long run and they should set their value functions to zero in this situation as well.

We have thus obtained the following value function for the preferences that a macro structure S_i has towards possible coalitions $\mathcal{C}$:

$$v_i(\mathcal{C}) = \begin{cases} \frac{r_i}{\rho(C)} & \text{if } i \in \mathcal{C} \text{ and } \rho(C) > \rho(D) \\ 0 & \text{otherwise} \end{cases} \quad \text{where}$$
$$\rho(C) = \sum_{j \in I_C} r_j \quad \text{and similarly} \quad \rho(D) = \sum_{j \notin I_C} r_j. \tag{14}$$

Since the values used in creating this function are fractions of control which is more than just an ordinal variable, actually, we believe that it is an interval variable in this setting, it follows that the so obtained function is not only an order value function, it is a measurable valued function. It remains to consider the overall outcome after all the structures have already analysed the situation and evaluated all the possible coalitions.

We know from the theory of social choice that there may be some controversy about creating the group preferences from the indivual ones. The well-known Arrow's paradox is showing that there is no general pattern of making rational, democratic, and fair group decisions on the basis of their individual preferences. However, in a more restricted model, where all the individuals have a common measure for expressing their preferences, there is a chance for that. The starting measure that the macro structures are using according to the above axioms is the fraction of their control in a long run and this has clearly the same meaning to all of them within our model.

We are now in position to use one of the results from the theory of social choice such as Theorem 8.2 (French, 1986), say, in order to obtain our main result.

Main Theorem. *In the model with additive resources the mesurable value functions $v_i(C)$ to measure the preference relation of a structure S_i towards coalitions C based on the Wald's maximin return criterion is given by (14). If v is any n-dimensional differentiable function with everywhere positive partial derivatives, then*

$$w(C) = v(v_1(C), v_2(C), \dots, v_n(C))$$

defines a fair order value function for the preferences of the group towards these coalitions.

We refer the reader to [Fre] for a detailed discussion of the meaning of the word "fair" in this theorem. Let us give a brief discussion of this result. A possible function v that comes naturally to one's mind might be

$$v(x_1, x_2, \dots, x_n) = \sum_{j=1}^{n} x_i^2.$$

Using (14) we get

$$w(C) = \begin{cases} \frac{\rho_2(C)}{\rho(C)^2} & \text{if } \rho(C) > \rho(D) \\ 0 & \text{otherwise} \end{cases} \quad \text{where}$$

$$\rho(C) = \sum_{j\in I_C} r_j, \quad \rho_2(C) = \sum_{j\in I_C} r_j^2, \quad \text{and} \quad \rho(D) = \sum_{j\notin I_C} r_j.$$

Thus, in order to predict the "right" coalition in power, we have to solve the following non-linear optimisation problem:

Find coalition C^* such that

$$w(C^*) = \max_{C} w(C)$$

subject to constraints

$$\rho(C) > \rho(D).$$

As an illustration take the main example from the previous section. It is clear that under the assumption $s + t > r$ we have that

$$w(\{1,2\}) = \frac{r^2 + s^2}{(r+s)^2}, \ w(\{1,3\}) = \frac{r^2 + t^2}{(r+t)^2}, \ w(\{2,3\}) = \frac{s^2 + t^2}{(s+t)^2},$$

$$\text{and} \ \ w(\{1,2,3\}) = \frac{r^2 + s^2 + t^2}{(r+s+t)^2}.$$

A straightforward consideration shows that coalition $\{2,3\}$ has the optimal value and is therefore preferred the most by these structures as we have already guessed in the previous section.

References

[1] Allen, G. D. (1992): Toward a dynamics for power and control in society. *J. Math. Sociology* **17**, 1-38.

[2] French, S. (1986): *Decision Theory: An Introduction to the Mathematics of Rationality.* Chichester: Ellis Horwood.

[3] Gantmacher, F. R. (1971): *The theory of matrices* (translated from Russian). New York: Chelsea.

[4] Omladič, M. and Omladič V. (1994): Hierarchical dynamics for power and control in society. *J. Math. Sociology* **18**, 293-313.

[5] Rothblum U. G. (1975): Algebraic eigenspaces of nonnegative matrices. *Lin. Alg. Appl.* **12**, 281-292.

[6] Schaefer, H. H. (1975): *Banach lattices of positive operators.* New York: Springer-Verlag.

[7] Schneider, H. (1986): The influence of the marked reduced graph of a nonnegative matrix on the Jordan form and on related properties: a survey. *Lin. Alg. Appl.* **84**, 161-189.

[8] Seneta, E. (1981): *Nonnegative matrices and Markov chains.* New York: Springer-Verlag.

[9] Zijm, W. H. M. (1983): *Nonnegative matrices in dynamic programming.* Amsterdam: Mathematisch Centrum.